Adaptive ILC for a class of nonlinear discrete-time systems in the presence of non-parametric uncertainties

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Abstract—In this paper, a discrete adaptive ILC is presented for a time-varying nonlinear system with nonparametric uncertainties. A novel estimation of nonparametric uncertainties is constructed just using the past I/O data, and the uncertainties are completely compensated such that the output tracking error is only affected by external disturbance. As a main contribution of this paper, all the discussions are based on the random initial condition and the iteration-varying target trajectories, which are the key obstacles of the existing ILC methods. The main result is shown by rigorous mathematical analysis.

I. INTRODUCTION

Iterative learning control [1] has been an intense area of research in the past two decades. To date, a lot of ILC schemes have been developed and widely applied [1-6]. Unlike some other intelligent control approaches, the effectiveness of ILC algorithms can be guaranteed via rigorous theoretical analysis.

Recent developments [7-12] in ILC show that adaptive ILC approach offers an appealing alternative to synthesize iterative learning controllers. In [7, 8], a standard adaptive ILC was proposed for uncertain robot manipulators, where the uncertain parameters are estimated along the time horizon, whereas the repetitive disturbances are compensated along the iteration horizon. However, as in standard adaptive control design, this technique requires the unknown system parameters to be constant. In [9-12], several ILC algorithms have been proposed based upon the use of a positive definite Lyapunov-like sequence which is made monotonically decreasing along the iteration axis via a suitable choice of the control input. In contrast to the standard adaptive control, this technique is shown to be able to handle systems with time-varying parameters since the adaptation law in this case is nothing else but a discrete integration along the iteration axis.

Again, these control laws depend on a certain a priori knowledge of the system dynamics, such as the parameterization of the system uncertainties, and require the identical initial condition and iteration-invariant reference trajectory, if the pointwise tracking performance is to be obtained.

It is worth pointing out that by making use of the analogy between the discrete time axis and the iterative learning axis, a new adaptive ILC scheme [13] was proposed by directly inherit the discrete-time adaptive control approach, and can achieve the almost perfect tracking performance over finite interval when both the strict initial state and the target trajectory vary iteratively. However, it still requires the parameterization of the system uncertainties, thus disables to address the non-parametric uncertainty as the typical ILC approaches proposed originally.

As a complementary methodology to the ILC, discrete-time adaptive control [14, 15] is a well established control strategy, and a number of adaptive schemes have been developed to deal with nonparametric model uncertainty [16, 17]. These methods mainly result in robust adaptive control and only eliminate the effect of the nonparametric uncertainty partially. In a recent work [18, 19], a new discrete-time adaptive control is explored by estimating the non-parametric uncertainties directly.

In this paper, we extend the well-established results in discrete-time adaptive control [18] to cope with the ILC tasks for a discrete-time nonlinear system with non-parametric uncertainties. The main idea is using the obtained I/O data to estimate and then to cancel the nonparametric uncertainty. The proposed adaptive ILC can guarantee the boundedness of all the signals, and further, the nonparametric uncertainty can be completely compensated such that the output tracking error is only affected by external disturbance.

The main contributions of the paper lie in that: 1) a novel estimation of nonparametric uncertainties is constructed just using the past input output data, 2) when the nonlinear function is generalized Lipschitz continuous and the external disturbance is iteration-varying within a bound, the robust property of the proposed adaptive ILC is guaranteed, 3) when the nonlinear function is standard Lipschitz continuous and the system is free of external disturbance, the proposed scheme guarantees an almost perfect tracking over the finite interval, and 4) all the discussions are done without requiring the identical conditions on the initial values and the target trajectories.

The rest of this paper is organized as follows. Section II is the problem formulation. Section III is the controller design,
and some preliminaries which will be needed in the proofs of the main theorem. The stability analysis is shown in Section IV. Finally Section V provides a summary with some concluding remarks.

II. PROBLEM FORMULATION

Consider the following system

\[ y_k(t+1) = f(y_k(t), t) + u_k(t) + w_k(t+1) \]  \hspace{1cm} (1)

where \( \{y_k(t)\}, \{u_k(t)\}, \) and \( \{w_k(t)\} \) are the system output, input and noise at time instant \( t \) of the \( k \)-th iteration, respectively. The nonlinear function \( f(\cdot): \mathbb{R}^1 \rightarrow \mathbb{R}^1 \) is completely unknown; \( t \in [0, 1, \cdots, T] \) is the finite time interval, and the subscript \( k = 1, 2, \cdots \) denotes the learning iterations.

In this system, two kinds of uncertainties exist: (i) Internal uncertainties are embodied in non-parametric part \( f(y_k(t), t) \); (ii) External uncertainties are embodied in the noise part \( w_k(t+1) \), which is iteration-varying.

To make further study, the following assumptions are used throughout this paper

**Assumption 1:** The unknown function \( f: \mathbb{R} \rightarrow \mathbb{R} \) belongs to the following uncertainty set

\[ F(L) = \{ f: |f(x) - f(y)| \leq L|x - y| + c \} \]  \hspace{1cm} (2)

where \( L \) is a positive constant, \( c \) is an arbitrary non-negative constant.

**Remark 1:** Assumption 1 is called generalized Lipschitz condition because it includes the standard Lipschitz condition as a special case \((c=0)\). Note that \( c \) is a quantity reflecting the possible discontinuity of the function \( f(\cdot) \).

**Assumption 2:** The noise sequence \( \{w_k(t)\} \) is bounded for all finite time interval \( t \in [0, 1, \cdots, T] \) and all iterations \( k \), i.e.,

\[ |w_k(t)| \leq w, \quad \forall t \in [0, 1, \cdots, T] \quad \text{and} \quad \forall k \]  \hspace{1cm} (3)

where \( w \) is an arbitrary positive constant.

**Assumption 3:** Since the target trajectory can be iteration-varying in this paper, we denote it at the \( k \)-th iteration as \( y_k^*(t) \) with bound \( S \), i.e.

\[ |y_k^*(t)| \leq S, \quad \forall t \in [0, 1, \cdots, T] \quad \text{and} \quad \forall k \]  \hspace{1cm} (4)

III. ADAPTIVE ITERATIVE LEARNING CONTROL DESIGN AND SOME AUXILIARY LEMMAS

A. Controller design

Let

\[ \tilde{h}_k(t) := \max_{0 \leq i \leq k} y_i(t) \quad \text{and} \quad h_k(t) := \min_{0 \leq i \leq k} y_i(t) \]  \hspace{1cm} (5)

and

\[ i_k := \arg \min_{0 \leq i \leq k} |y_k(t) - y_i(t)| \]

i.e.,

\[ |y_k(t) - y_{i_k}(t)| = \arg \min_{0 \leq i \leq k} |y_k(t) - y_i(t)|. \]  \hspace{1cm} (6)

At any learning iteration \( k \geq 1 \), the estimate of \( f(y_k(t), t) \) is defined as

\[ \hat{f}_k(y_k(t), t) = y_k(t+1) - u_k(t) \]  \hspace{1cm} (7)

which can be rewritten as

\[ \hat{f}_k(y_k(t), t) = f(y_k(t), t) + w_k(t+1), \quad k \geq 1 \]  \hspace{1cm} (8)

We remark that the estimator (6) and (7) may be referred to as the nearest neighbor (NN) estimator for \( f(\cdot) \) as can be seen intuitively from (8). It is a natural one when we only know the generalized Lipschitz continuity of \( f(\cdot) \) and the boundedness of the noises \( \{w_k(t)\} \).

Denote

\[ u_k(t) = -\hat{f}_k(y_k(t), t) + \frac{1}{2} (h_k + \tilde{h}_k(t)), \quad k \geq 1 \]  \hspace{1cm} (9)

\[ u_k^* = -\hat{f}_k(y_k(t), t) + y_k^*(t), \quad k \geq 1 \]  \hspace{1cm} (10)

Then the adaptive iterative learning control law is defined as

\[ u_k(t) = \begin{cases} u_k(t), & \text{if } |y_k(t) - y_i_k(t)| > \varepsilon \\ u_k^*(t), & \text{if } |y_k(t) - y_i_k(t)| \leq \varepsilon \end{cases} \]  \hspace{1cm} (11)

where \( \varepsilon > 0 \) can be chosen arbitrarily.

B. Preliminaries

In this subsection, we present two auxiliary lemmas which will be needed in the proofs of the main theorem stated in the last section.

**Lemma 1:** Let \( L < 3/2 + \sqrt{2} \), \( d \geq 0 \) and \( n_0 \geq 0 \) be constants. If a nonnegative sequence \( \{h_n, n \geq 0\} \) satisfies

\[ h_{n+1} \leq \left( L \max_{0 \leq i \leq n} h_i - \frac{1}{2} \sum_{i=0}^{n} h_i + d \right)^+, \quad \forall n \geq n_0 \]  \hspace{1cm} (12)

where \( (x)^+ = \max\{0, x\}, \forall x \in \mathbb{R}^1 \), then

\[ \lim_{n \rightarrow \infty} \sum_{i=0}^{n} h_i < \infty. \]  \hspace{1cm} (13)

The proof of Lemma 1 can be seen in [18].

**Lemma 2:** If a sequence \( \{z_n, n \geq 0\} \) is bounded, i.e.,

\[ |z_n| \leq M < \infty, \quad \forall n \geq 0, \quad \text{then} \]

\[ \lim_{n \rightarrow 0} \sum_{i=0}^{n} z_i = 0 \]  \hspace{1cm} (14)

where

\[ i_n = \arg \min_{0 \leq i \leq k} |z_i - z_n| \]  \hspace{1cm} (15)

The proof of Lemma 2 can be seen in [18].

IV. STABILITY ANALYSIS

**Theorem 1:** For any \( f \in F(L) \) with \( L \leq 3/2 + \sqrt{2} \), the adaptive ILC (5)-(11) for the corresponding system (1) can
guarantee the following tracking error is bounded for all iterations \( k \) over the entire finite time interval \( t \in \{0,1,\cdots,T\} \).

More precisely, we have
\[
\lim_{k \to \infty} |y_k(t) - y^*_k(t)| \leq c + 2w, \quad \forall t \in \{0,1,\cdots,T\}
\] (16)
where \( c \) is defined in (2) with \( \gamma > 0 \) chosen to satisfy \( L_1 = L + \gamma < 3/2 + \sqrt{2} \), and \( w \) is defined in (3).

**Corollary 1:** If \( L < 3/2 + \sqrt{2} \), \( c = w = 0 \), then the adaptive ILC (5)-(11) can guarantee the tracking error converges to zero pointwisely over the finite time interval \( \{1,2,\cdots,T\} \) as \( k \) approaches to infinity, i.e.
\[
\lim_{k \to \infty} |y_k(t) - y^*_k(t)| = 0, \quad \forall t \in \{1,2,\cdots,T\}
\] (17)

**Corollary 2:** If \( L < 3/2 + \sqrt{2} \), \( c = 0 \), and the external uncertainty \( u(t+1) = w(t+1) = 0 \), the adaptive ILC (5)-(11) still can guarantee
\[
\lim_{k \to \infty} |y_k(t) - y^*_k(t)| = 0, \quad \forall t \in \{1,2,\cdots,T\}
\] (18)

**Remark 2:** Note that we assume the initial state \( y_0(0) \), cannot be manipulated via any control signals. Thus the initial error \( e_0 = y_0(0) - y^*_0(0) \), shall be excluded from the learning control objective.

**Proof:** First, we introduce some notations. Denote
\[
B_k(t) := \{y_k(t), \delta_k(t)\}, \quad \Delta B_k(t) := B_k(t) - B_{k-1}(t)
\] (19)
and
\[
\|\mathcal{B}_k(t)\| := \|\delta_k(t) - \bar{\delta}_k(t)\|, \quad \|\Delta B_k(t)\| := \|B_k(t) - \|B_{k-1}(t)\|
\] (20)
where \( \Delta B_k(t) := B_k(t) \), \( \delta_k(t) \) and \( \bar{\delta}_k(t) \) are defined in (5).

By the definition (5), we have
\[
\bar{\delta}_k(t) \geq \delta_{k+1}(t), \quad \bar{\delta}_k(t) \leq \delta_{k-1}(t)
\]
and
\[
\delta_k(t) - \delta_{k-1}(t) = \bar{\delta}_k(t) - \delta_{k-1}(t) = 0,
\]
thus, the interval sequence \( \{\delta_k(t), k \geq 0\} \) is nondecreasing and the vector \( \Delta B_k(t) \) (an interval sequence) can be a null set \( \Phi \) and
\[
B_k(t) = \bigcup_{i=0}^{k} \Delta B_i(t) \quad \text{and} \quad \Delta B_k(t) \cap \Delta B_j(t) = \Phi, \quad i \neq j.
\] (21)

**Step 1:** We analyze some properties of the notations (19), and (20).

First, it is clear that
\[
\begin{cases}
\|B_{k+1}(t)\| = \|B_k(t)\|, & \text{if } y_{k+1}(t) \in B_k(t) \\
\|B_{k+1}(t)\| = \|y_{k+1}(t) - \frac{1}{2} \|\mathcal{B}_k(t) + \bar{\delta}_k(t)\| + \frac{1}{2} \|B_k(t)\|, & \text{if } y_{k+1}(t) \notin B_k(t)
\end{cases}
\] (22)

Then since
\[
\left| y_{k+1}(t) - \frac{1}{2} \|\mathcal{B}_k(t) + \bar{\delta}_k(t)\| \right| > \frac{1}{2} \|B_k(t)\| \iff y_{k+1}(t) \notin B_k(t),
\]
we have
\[
\|B_{k+1}(t)\| = \max \left\{ \|y_{k+1}(t) - \frac{1}{2} (\mathcal{B}_k(t) + \bar{\delta}_k(t))\| + \frac{1}{2} \|B_k(t)\|, \|B_k(t)\| \right\}
\] (23)

Now we proceed to prove that
\[
\left| y_k(t) - y^*_k(t) \right| \leq \max_{0 \leq k \leq k} \|\Delta B_k(t)\|, \quad \forall k \geq 1 \quad \text{and} \quad t \in \{0,1,\cdots,T\}
\] (24)
where \( i_k \) is defined in (6). We consider two cases separately.

**Case (1):** If \( y_k(t) \notin B_{k-1}(t) \), then by definitions (5), (6), (19), and (20), we have
\[
\left| y_k(t) - y^*_k(t) \right| = \left| B_k(t) - B_{k-1}(t) \right| = \|\Delta B_k(t)\|
\]
**Case (2):** If \( y_k(t) \notin B_{k-1}(t) \), then by (21), we know \( y_k(t) \in \Delta B_k(t) \) for some \( 0 \leq i \leq k-1 \). Then by (6) we have
\[
\left| y_k(t) - y^*_k(t) \right| \leq \|\Delta B_i(t)\|, \quad \text{for some } i.
\]
Combining the two cases above, we see that (24) is true.

**Step 2:** We proceed to find a recursive inequality on \( \|\Delta B_k(t)\|, k \geq 0 \).

By (1) and (9)-(11), we have
\[
y_k(t+1) = f(y_k(t),t) - \hat{f}_k(y_k(t),t) + \frac{1}{2} \|\mathcal{B}_k(t) + \bar{\delta}_k(t)\| + w_k(t+1), \quad \text{if } \left| y_k(t) - y^*_k(t) \right| > \varepsilon
\]
\[
y_k(t+1) = f(y_k(t),t) - \hat{f}_k(y_k(t),t) + \frac{1}{2} \|\mathcal{B}_k(t) + \bar{\delta}_k(t)\| + w_k(t+1), \quad \text{if } \left| y_k(t) - y^*_k(t) \right| \leq \varepsilon
\]
Hence by (8), we know that if \( \left| y_k(t) - y^*_k(t) \right| > \varepsilon \), then
\[
y_k(t+1) = f(y_k(t),t) - f(y^*_k(t),t) + \frac{1}{2} \|\mathcal{B}_k(t) + \bar{\delta}_k(t)\| - w_k(t+1), \quad \text{if } \left| y_k(t) - y^*_k(t) \right| > \varepsilon
\]
and if \( \left| y_k(t) - y^*_k(t) \right| \leq \varepsilon \), then
\[
y_k(t+1) = f(y_k(t),t) - f(y^*_k(t),t) + \frac{1}{2} \|\mathcal{B}_k(t) + \bar{\delta}_k(t)\| - w_k(t+1), \quad \text{if } \left| y_k(t) - y^*_k(t) \right| \leq \varepsilon
\]

Now, if for some \( k \geq 1 \), \( \left| y_k(t) - y^*_k(t) \right| > \varepsilon \), then by (23) and (25),
\[
\|B_{k+1}(t)\| \leq \max \left\{ \|y_{k+1}(t) - \frac{1}{2} \|\mathcal{B}_k(t) + \bar{\delta}_k(t)\| + \frac{1}{2} \|B_k(t)\|, \|B_k(t)\| \right\}
\]
\[
\|B_{k+1}(t)\| \leq \max \left\{ \|y_{k+1}(t) - \frac{1}{2} \|\mathcal{B}_k(t) + \bar{\delta}_k(t)\| + \frac{1}{2} \|B_k(t)\|, \|B_k(t)\| \right\}
\]
Hence by (3) and the definition of \( \|\mathcal{B}_k(t)\| \), we have
\[
\|B_k(t+1)\| \leq \|B_{k+1}(t)\| \leq \max \left\{ \|f(y_k(t),t) - f(y^*_k(t),t)\| + 2w + \frac{1}{2} \|B_k(t+1)\| \right\}
\]
Furthermore, since \( f \in F(L) \) and \( L < 3/2 + \sqrt{2} \), there exist some \( L_0 \) satisfying \( L < L_0 < 3/2 + \sqrt{2} \) and some \( c \geq 0 \) such that \( f \in F(L_1,c) \). Hence,
\[ |y_k(t+1)| \leq |y_{k+1}(t+1)| \]
\[ \leq \max \left\{ L_1 |y_k(t) - y_{i_k}(t)| + c + 2w - \frac{1}{2} B_k(t+1) \right\} \]
Consequently, by (20), we have

\[ 0 \leq |\Delta B_{k+1}(t+1)| \leq \left( L_1 \max \left\{ |\Delta B_k(t)| + c + 2w - \frac{1}{2} \sum_{i=0}^k |\Delta B_i(t)| \right\} \right)^+ \]

which holds for any \( k \geq 1 \) with \( y_k(t) - y_{i_k}(t) > \varepsilon \).

**Step 3:** We now prove that for any \( s \geq 0 \), there exists some \( \tau > s \) such that \( |y_{\tau}(t) - y_{i_{\tau}}(t)| \leq \varepsilon \). Otherwise, if

\[ |y_k(t) - y_{i_k}(t)| > \varepsilon, \quad \forall k > s \]
then by (28), \( \forall k > s \)

\[ 0 \leq |\Delta B_{k+1}(t)| \leq \left( L_1 \max \left\{ |\Delta B_k(t)| + c + 2w - \frac{1}{2} \sum_{i=0}^k |\Delta B_i(t)| \right\} \right)^+ \]

Then by Lemma 1, \( \lim_{k \to \infty} \sum_{i=0}^k |\Delta B_i(t)| < \infty \), i.e.,

\[ \lim_{k \to \infty} B_k(t) < \infty. \]

Thus by the definition of \( \lim_{k \to \infty} B_k \), we have \( |y_k(t)| < \infty \). Consequently, by Lemma 2, we have \( \lim_{k \to \infty} |y_k(t) - y_{i_k}(t)| = 0 \), which contradicts to (30). Hence, there exist \( \tau > s \) such that \( |y_{\tau}(t) - y_{i_{\tau}}(t)| \leq \varepsilon \). Therefore by (26), we have for \( f \in F(L_1,c) \)

\[ |y_{\tau}(t+1)| \leq |f(y_{\tau}(t),t) - f(y_{i_{\tau}}(t),t)| + |y_{\tau}'(t+1)| + 2w \]
\[ \leq L_1 |y_{\tau}(t) - y_{i_{\tau}}(t)| + c + S + 2w \]
\[ \leq L_1 \varepsilon + c + S + 2w \]
From which we arrive at (29) by setting \( \tau = \tau + 1 \).

**Step 4:** We prove the boundedness of the whole sequence \( \{y_k(t), k \geq 0\} \).

Define

\[ k_0 = \inf \{ k : |y_k(t)| \leq L_1 \varepsilon + c + S + 2w \} \]
\[ k_n = \inf \{ k : |y_k(t)| \leq L_1 \varepsilon + c + S + 2w \}, n \geq 1 \]
Then by (29), we have \( k_n < \infty, \forall n \geq 0 \).

Let \( z_n(t) = y_{k_n}(t) \). By (32), we have

\[ |z_n(t)| \leq L_1 \varepsilon + c + S + 2w, \forall n \geq 0. \]

Then by Lemma 2 with \( i_n \) defined in (15), we have

\[ \lim_{n \to \infty} |z_n(t) - y(t)| = 0. \]

Thus there exists some \( n_0 \geq 0 \) such that for all \( n \geq n_0 \)

\[ |z_n(t) - y(t)| \leq \varepsilon, \quad \text{i.e.,} \quad \min_{0 \leq s < n} |z_n(t) - y(t)| \leq \varepsilon, \]

Consequently, for any \( n \geq n_0 \)

\[ |y_{k_n}(t) - y_i(t)| = \min_{0 \leq s < n} |y_{k_n}(t) - y_i(t)| \leq \varepsilon \]

where \( i_{k_n} = \arg \min_{0 \leq s < n} |y_{k_n}(t) - y_i(t)| \).

Hence by (26) we have for any \( f \in F(L_1,c) \)

\[ |y_{k_n}(t+1)| \leq |f(y_{k_n}(t),t) - f(y_{i_{k_n}}(t),t) + y_{k_n}'(t+1)| + 2w \]
\[ \leq L_1 \varepsilon + c + S + 2w, \forall n \geq n_0 \]
So by (32), \( k_{n+1} = k_n + 1, \forall n \geq n_0 \)
which implies that

\[ |y_k(t)| \leq L_1 \varepsilon + c + S + 2w, \forall k \geq k_{n_0}. \]

**Step 5:** Finally, we give an upper bound for the asymptotic tracking error.

From (33), by Lemma 2 again, we have

\[ \lim_{k \to \infty} |y_k(t) - y_i(t)| = 0. \]

Hence there exists some \( K > 0 \) such that

\[ |y_k(t) - y_i(t)| \leq \varepsilon, \quad \forall k \geq K. \]

Finally, by (26) and (34), we have for \( k \geq K \)

\[ |y_{k}(t+1) - y_{i_{k}}(t+1)| \]
\[ \leq |f(y_{k}(t),t) - f(y_{i_{k}}(t),t) + y_{k}'(t+1)| + 2w \]
\[ \leq L_1 |y_{k}(t) - y_{i_{k}}(t)| + c + S + 2w \rightarrow c + 2w, \text{ as } k \to \infty \]
which is a tantamount of Theorem 2. Hence the proof is completed.

**Remark 3:** Corollary 1 is apparent. And Corollary 2 can be easily proved by set \( w_k(t+1) = w(t+1) \) for all iterations, which shows that ILC can completely eliminate the effect of repeatable external uncertainties imposed on the tracking performance.

V. CONCLUSION

In this paper, a novel adaptive ILC has been synthesized for a nonlinear discrete-time system with nonparametric uncertainties, where the nonlinear function is generalized Lipschitz continuous and the external disturbances are iteration-varying. The adaptive ILC guarantees the boundedness of all the close-loop signals and completely compensates the uncertainties. The system output exactly
tracks the iteration-varying target trajectory under random initial conditions in the absence of external disturbance (or external disturbance is repeatable) if the nonlinear function is standard Lipschitz continuous.

REFERENCES