Abstract—In this paper we develop a new approach to iterative learning control for the practically relevant case of deterministic discrete linear plants where the first Markov parameter is zero. The basis for this is a 2D systems approach that, by using a strong form of stability for linear repetitive processes, also allows us to consider both trial-to-trial and along the trial performance. This is in contrast to many other approaches where the sole emphasis is on error convergence. The resulting design computations are in terms of Linear Matrix Inequalities (LMIs). Results from experimentally applying the resulting control law to one axis of a gantry robot are also given.

I. INTRODUCTION

Iterative learning control (ILC) is a technique for controlling systems operating in a repetitive, or trial-to-trial, mode with the requirement that a reference trajectory $r(t)$ defined over a finite interval $0 \leq t \leq \alpha$, where $\alpha$ denotes the trial length, is followed to a high precision. Examples of such systems include robotic manipulators that are required to repeat a given task to high precision, chemical batch processes, also allows us to consider both trial-to-trial and along the trial performance. In particular, that trial-to-trial convergence occurs but produces along the trial performance. In previous work [4], it has been shown that fast convergence could lead to unsatisfactory performance. In this paper we extend the 2D systems results in [4] to this case.

In this paper, the null and identity matrices with the required dimensions are denoted by 0 and $I$ respectively. Also $\Gamma > 0$ and $\Gamma < 0$ respectively are used to denote symmetric matrices which are positive definite and negative definite, respectively. The symbol $r(\cdot)$ is used to denote the spectral radius of a given matrix, that is, if $H$ is an $n \times n$ matrix then $r(H) = \max_{1 \leq i \leq n} |\lambda_i|$ where $\lambda_i$ is an eigenvalue of $H$.

II. BACKGROUND AND INITIAL ANALYSIS

The plants considered in this paper are assumed to be adequately represented by discrete linear time-invariant systems described by the state-space triple $\{A, B, C\}$. In an ILC setting for linear time-invariant dynamics, the state-space model is written as

$$x_k(p + 1) = Ax_k(p) + Bu_k(p), \quad p = 0, 1, \ldots, \alpha - 1$$
$$y_k(p) = Cx_k(p)$$

where on trial $k x_k(p) \in \mathbb{R}^n$ is the state vector, $y_k(p) \in \mathbb{R}^m$ is the output vector, $u_k(p) \in \mathbb{R}^r$ is the vector of control inputs, and the trial length $\alpha < \infty$. If the signal to be tracked is denoted by $y_{ref}(p)$ then $e_k(p) = y_{ref}(p) - y_k(p)$ is the error on trial $k$. The most basic requirement now is control law design to force trial-to-trial error convergence (that is, in the $k$ direction).

As discussed in the previous section of this paper, design for trial-to-trial error convergence only may conflict with along the trial performance. In particular, that trial-to-trial convergence occurs but produces along the trial performance which is far from satisfactory for many practical applications, e.g. a gantry robot whose task is to collect an object from a
location, place it on a moving conveyor, and then return for the next one and so on. If, for example, the object has an open top and is filled with liquid, and/or is fragile in nature, then unwanted vibrations during the transfer time could have very detrimental effects. Hence in such cases there is also a need to control the along the trial dynamics and in this paper the method is to use a stronger form of stability theory for linear repetitive processes, which are one class of 2D linear systems. Next we introduce the required background on linear repetitive processes.

The unique characteristic of a repetitive, or multipass [5], process is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. This, in turn, leads to the unique control problem in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass-to-pass direction.

To introduce a formal definition, let \( \alpha < +\infty \) denote the pass length (assumed constant). Then in a repetitive process the pass profile \( y_k(p) \) generated on pass \( k \) acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile \( y_{k+1}(p) \), \( p = 0, 1, \ldots, \alpha - 1, k \geq 0 \).

Attempts to control these processes using standard (or 1D) systems theory and algorithms fail (except in a few very restrictive special cases) precisely because such an approach ignores their inherent 2D systems structure, i.e. information propagation occurs from pass-to-pass (\( k \) direction) and along a given pass (\( t \) direction) and also the initial conditions are reset before the start of each new pass. To remove these deficiencies, a rigorous stability theory has been developed [5] based on an abstract model of the dynamics in a Banach space setting which includes a very large class of processes with linear dynamics and a constant pass length as special cases, including those described by (2) below. In terms of their dynamics, it is the pass-to-pass coupling (noting again their unique feature) which is critical. This is of the form \( y_{k+1} = L_\alpha y_k \), where \( y_k \in E_\alpha \) (\( E_\alpha \) a Banach space with norm \( \| \cdot \| \)) and \( L_\alpha \) is a bounded linear operator mapping \( E_\alpha \) into itself.

The most basic discrete linear repetitive process state-space model has the following form over \( p = 0, 1, \ldots, \alpha - 1, k \geq 0 \)

\[
\begin{align*}
x_{k+1}(p+1) &= \hat{A}x_{k+1}(p) + \hat{B}u_{k+1}(p) + \hat{B}_0 y_k(p) \\
y_{k+1}(p) &= \hat{C}x_{k+1}(p) + \hat{D}u_{k+1}(p) + \hat{D}_0 y_k(p)
\end{align*}
\] (2)

where \( x_k(p) \in \mathbb{R}^n, u_k(p) \in \mathbb{R}^r, y_k(p) \in \mathbb{R}^m \) are the state, input and pass profile vectors respectively. The boundary conditions \( x_k(0) \) and \( y_0(p) \) are assumed to be constant vectors of appropriate dimension.

To construct an ILC scheme for the process of (1) introduce

\[
\begin{align*}
\eta_{k+1}(p) &= x_{k+1}(p) - x_k(p) - 1 \\
\Delta u_{k+1}(p) &= u_{k+1}(p) - u_k(p)
\end{align*}
\] (3)

Then we have

\[
\eta_{k+1}(p + 1) = \hat{A}\eta_{k+1}(p) + B\Delta u_{k+1}(p - 1)
\] (4)

Consider also a control law of the form

\[
\Delta u_{k+1}(p) = K_1\eta_{k+1}(p + 1) + K_2\Delta u_k(p + 1)
\] (5)

and hence

\[
\eta_{k+1}(p + 1) = (A + BK_1)\eta_{k+1}(p) + BK_2\Delta u_k(p)
\] (6)

Noting that \( e_{k+1}(p) - e_k(p) = y_k(p) - y_{k+1}(p) \) now gives

\[
e_{k+1}(p) - e_k(p) = CA(x_k(p - 1) - x_{k+1}(p - 1)) + CB(u_k(p - 1) - u_{k+1}(p - 1))
\] (7)

and using (3)

\[
e_{k+1}(p) - e_k(p) = -CA\eta_{k+1}(p) - CB\Delta u_{k+1}(p - 1)
\]

or, on use of (5),

\[
e_{k+1}(p) = -C(A + BK_1)\eta_{k+1}(p) + (I - CBK_2)e_k(p)
\] (8)

Hence (6) and (8) can be written as

\[
\begin{align*}
\eta_{k+1}(p + 1) &= \hat{A}\eta_{k+1}(p) + \hat{B}_0 e_k(p) \\
e_{k+1}(p) &= \hat{C}\eta_{k+1}(p) + \hat{D}_0 e_k(p)
\end{align*}
\] (9)

where

\[
\begin{align*}
\hat{A} &= A + BK_1 \\
\hat{B}_0 &= BK_2 \\
\hat{C} &= -C(A + BK_1) \\
\hat{D}_0 &= I - CBK_2
\end{align*}
\] (10)

which is of the form (2) and hence the repetitive process stability theory can be applied to this ILC control scheme.

The stability theory for linear repetitive processes with constant pass length consists of two distinct concepts. Asymptotic stability, i.e. bounded-input bounded-output (BIBO) stability over the fixed finite pass length \( \alpha > 0 \), requires the existence of finite real scalars \( M, \lambda \) such that \( \| K_n \| \leq M, \lambda_n \geq \lambda, k \geq 0 \) (where \( \| \cdot \| \) also denotes the induced operator norm). For processes described by (2) it has been shown elsewhere (see, for example, Chapter 3 of [5]) that this property holds if, and only if, \( r(\hat{D}_0) < 1 \). When applied to the ILC state-space model (9) this requires that \( r(\hat{D}_0) = r(I - CBK_2) < 1 \).

This last condition is precisely that obtained by applying 2D discrete linear systems stability theory to (9) as first proposed in [6] to ensure trial-to-trial error convergence only. It is easy to construct examples where \( r(\hat{D}_0) < 1 \) but the performance along the trial is very poor. (The source of this problem is that this condition demands that the trial output is bounded over a finite duration and an unstable linear system can only produce a bounded output). In repetitive process stability theory, asymptotic stability guarantees that the sequence of pass profiles generated by an example with this property converges strongly as \( k \to \infty \) to a so-called limit profile whose dynamics for the processes considered here can be obtained by letting \( k \to \infty \) in the state-space
model. This results in a 1D discrete linear systems state-space model but it is possible that its state matrix will be unstable. Stability along the pass prevents this from happening by demanding the BIBO stability property for all possible values of the finite pass length. This is the route by which design taking account of trial-to-trial error convergence and along the trial dynamics can be undertaken, where details for one case with experimental verification can be found in [4].

One common problem in many ILC designs for linear systems arises when $CB = 0$ and hence the condition $r(\bar{D}_0) < 1$ cannot be satisfied. This paper develops a method to overcome this problem and the resulting control law is then applied to the model of one axis of a gantry robot discussed in the previous section.

III. ANALYSIS

Consider an example with $CB = 0$. Then application of (5) gives

$$
\eta_{k+1}(p+1) = A\eta_{k+1}(p) + B\Delta u_{k+1}(p-1) - CAB\eta_{k+1}(p) + e_k(p)
$$

or, on substituting the first equation into the term $-CAB\eta_{k+1}(p)$ in the second equation and some routine manipulations,

$$
\eta_{k+1}(p+1) = A^2\eta_{k+1}(p-1) + B\Delta u_{k+1}(p-1) + AB\Delta u_{k+1}(p-2) - CAB^2\eta_{k+1}(p-1) - CAB\Delta u_{k+1}(p-2) + e_k(p)
$$

Application of the modified control law

$$
\Delta u_{k+1}(p) = K_1\eta_{k+1}(p+1) + K_2e_k(p+2)
$$

now gives

$$
\eta_{k+1}(p+1) = A^2\eta_{k+1}(p-1) + B(K_1\eta_{k+1}(p) + K_2e_k(p+1)) + AB(K_1\eta_{k+1}(p-1) + K_2e_k(p)) - CAB^2\eta_{k+1}(p-1) - CAB(K_1\eta_{k+1}(p-1) + K_2e_k(p)) + e_k(p)
$$

(12)

Now introduce

$$
\tilde{\eta}_{k+1}(p) = \begin{bmatrix} \eta_{k+1}(p) \\ \eta_{k+1}(p-1) \end{bmatrix}
$$

and hence (14) can be written as

$$
\tilde{\eta}_{k+1}(p+1) = \tilde{A}\tilde{\eta}_{k+1}(p) + \tilde{B}_1e_k(p+1) + \tilde{B}_0e_k(p)
$$

(16)

where

$$
\tilde{A} = \begin{bmatrix} BK_1 \ (A^2 + ABK_1) \\ I \ 0 \end{bmatrix}
$$

$$
\tilde{B}_1 = \begin{bmatrix} BK_2 \\ 0 \end{bmatrix}
$$

$$
\tilde{B}_0 = \begin{bmatrix} ABK_2 \\ 0 \end{bmatrix}
$$

$$
\tilde{C} = -\begin{bmatrix} 0 \\ (CA^2 + CABK_1) \end{bmatrix}
$$

$$
\tilde{D}_1 = 0
$$

$$
\tilde{D}_0 = (I - CABK_2)
$$

(17)

Hence if $CAB \neq 0$, we recover the ability to influence the pass to pass error dynamics. If, however, $CAB = 0$ we can continue this procedure until we reach the first non-zero Markov parameter. Note, however, that when $CA'B$ is the first non-zero Markov parameter we lose the possibility to control the first $r$ points on each trial. Moreover, due to the presence of the term $e_k(p+1)$ in the first equation of (16) this repetitive process is not of the form (2) and hence the control theory for this model cannot be applied. It is, however, an example of a so-called ‘wave’ discrete linear repetitive process (see [7]) and next we give the required background necessary to continue with the ILC analysis.

A wave discrete linear repetitive process [7] is described by the following state-space model over $p = 0, 1, \ldots, \alpha - 1$, $k \geq 1$,

$$
x_{k+1}(p+1) = Ax_{k+1}(p) + \sum_{i=-\gamma}^{\gamma} B_iy_k(p+i)
$$

$$
y_{k+1}(p) = Cx_{k+1}(p) + \sum_{i=-\bar{\gamma}}^{\bar{\gamma}} D_iy_k(p+i)
$$

(18)

where $\gamma$ and $\bar{\gamma}$ are some natural numbers and the rest of notation is the same as that for (2). (The model of (16) is recovered on setting $\gamma = 0$ and $\bar{\gamma} = 1$ in this last state-space model).

With the overall aim of controlling both trial-to-trial error convergence and along the trial performance we use a Lyapunov function interpretation of stability along the trial (for a similar approach in the case of wave discrete linear repetitive processes see [7]). Consider, therefore, the following candidate Lyapunov function

$$
V(k, p) := V_1(k, p) + V_2(k, p)
$$

(19)

where

$$
V_1(k, p) = \eta_{k+1}^T(p)P_0\eta_{k+1}(p)
$$

$$
V_2(k, p) = e_k^T(p+1)P_1e_k(p+1) + e_k^T(p)P_0e_k(p)
$$

and $P_{00} > 0$, $P_1 > 0$, $P_0 > 0$. Note that this function is the combination of two independent indeterminates due to the 2D structure of repetitive processes with increment

$$
\Delta V(k, p) := \Delta V_1(k, p) + \Delta V_2(k, p)
$$

(20)
where
\[
\Delta V_1(k, p) = \eta_{k+1}^T(p+1) P_{00} \eta_{k+1}(p+1) - \eta_{k+1}^T(p) P_{00} \eta_{k+1}(p) - [e_k^T(p+1) e_k^T(p)] \text{diag}(P_1, P_0) \begin{bmatrix} e_k(p+1) \\ e_k(p) \end{bmatrix}
\]

Following the analysis in [7], we have that (18) is stable along the trial if
\[
\Delta V(k, p) < 0
\]
for all non-zero \(\eta_{k+1}(p)\) and \(e_k(p)\).

Introduce the notation
\[
\Phi = \begin{bmatrix} \tilde{A} & \tilde{B}_0 & \tilde{B}_1 \\ \tilde{C} & \tilde{D}_0 & \tilde{D}_1 \end{bmatrix}
\]
and
\[
\theta = \begin{bmatrix} 0 & P_{00} & 0 \\ 0 & 0 & P_0 \\ 0 & 0 & P_1 \end{bmatrix}
\]
Then we have the following result whose proof follows as an obvious extension of results in [7] and is hence omitted here.

**Theorem 1:** The wave repetitive process of (16) is stable along the trial if
\[
\Phi^T \theta \Phi - \theta < 0
\]

The following result solves the ILC design problem considered here.

**Theorem 2:** An ILC scheme which can be written in the form (16) is stable along the trial if there exist matrices \(X_1 > 0\) and \(X_2 > 0\) such that the LMI
\[
\begin{bmatrix}
-X_1 & 0 & 0 \\
0 & -X_1 & 0 \\
0 & 0 & -X_1 \\
BN_1 & A^T X_1 + ABN_1 & ABN_2 \\
X_1 & 0 & 0 \\
0 & -CA^2 X_1 - CABN_1 & X_2 - CABB_2 \\
0 & -CA^2 X_1 - CABN_1 & X_2 - CABB_2 \\
0 & N_2^T B^T & X_1 \\
-\eta^T e_k + 1 & N_2^T B^T A^T & 0 \\
0 & -X_2 & N_2^T B^T \\
BN_2 & -X_1 & 0 \\
0 & 0 & -X_1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} < 0
\]
(22)

If this condition holds, the control law matrices \(K_1\) and \(K_2\) of (16) can be computed from
\[
K_1 = N_1 X_1^{-1}, \quad K_2 = N_2 X_2^{-1}
\]

**Proof:** Application of the Schur’s complement formula to (21) yields
\[
\begin{bmatrix}
-P_{00} & 0 & 0 & \tilde{A}^T & \tilde{D}_0^T & \tilde{D}_1^T \\
0 & -P_0 & 0 & \tilde{B}_0^T & \tilde{D}_0^T & \tilde{D}_1^T \\
0 & 0 & -P_1 & \tilde{B}_1^T & \tilde{D}_1^T & \tilde{D}_1^T \\
\tilde{A} & \tilde{B}_0 & \tilde{B}_1 & -P_{00}^{-1} & 0 & 0 \\
\tilde{C} & \tilde{D}_0 & \tilde{D}_1 & 0 & -P_{00}^{-1} & 0 \\
\tilde{C} & \tilde{D}_0 & \tilde{D}_1 & 0 & 0 & -P_{00}^{-1}
\end{bmatrix} \prec 0
\]
(24)

Also applying obvious congruence transforms and introducing
\[
P_{00}^{-1} = \begin{bmatrix} X_1 & 0 \\ 0 & X_1 \end{bmatrix}, \quad P_{01}^{-1} = X_2, \quad P_{11}^{-1} = X_2
\]
(25)
yields the result (after application of routine algebraic manipulations which are omitted here).
IV. Experimental Results

To experimentally verify the practical value of the described approach tests were undertaken using a multi-axis gantry robot, see Fig. 1. The axes marked in this figure and this system has previously been used for testing and comparing the performance of other ILC algorithms, see, for example, [8]. Each axis of the gantry robot is controlled individually and the models of all were obtained by means of frequency response tests that determined the continuous-time transfer-functions.

The gantry robot consists of three axes and is designed to simulate a pick and place task. A desired 3D trajectory is shown in Fig. 2. The transfer-function for the $Z$-axis is

$$G_z(s) = \frac{15.8869(s + 850.3)}{s(s^2 + 707.6s + 3.377 \times 10^5)}$$  \hspace{1cm} (26)

and discretization using the zero-order hold method with a sampling time of $T_s = 0.01$ sec gives the $z$ transfer-function

$$G_z(z) = \frac{0.00036482(z^2 + 0.09791z + 0.005951)}{(z - 1)(z^2 + 0.005922z + 0.0008451)}$$  \hspace{1cm} (27)

The $Z$-axis component of the 3D reference trajectory with 200 samples (hence $\alpha = 200$ is shown in Fig. 3. and the experimentally measured trial-to-trial error $e_k(p) = y_{ref}(p) - y_k(p)$ over 200 trials is shown in Fig. 4. The control input sequence is acceptable for implementation and the trial outputs $y_k(p)$ in Fig. 5 Finally, the mean squared error over 200 trials is shown in Fig. 6.

A one sample delay is produced by the action of a zero-order hold, contained within the real-time control card, being fed to the differential equation describing the plant. Hence for design we replace $G_z(z)$ by

$$\hat{G}_z(z) = \frac{0.00036482(z^2 + 0.09791z + 0.005951)}{z(z - 1)(z^2 + 0.005922z + 0.0008451)}$$  \hspace{1cm} (27)

and the corresponding state-space model matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.0008 & -0.0030 & 1 \\ 0 & 0 & 0.0008 & -0.0030 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$
$$B = [0 \ 0 \ 0 \ 0.0313]^T$$ \hspace{1cm} (28)
$$C = [0.0001 \ 0.0122 \ 0.0117 \ 0]$$

In this state-space model $CB = 0$ and $CAB = 0.0003648$.

Solving the LMI of (22) and using (23) gives the control law matrices

$$K_1 = \begin{bmatrix} 0.8164 & -5.028 & -6.691 & -5.315 \end{bmatrix}$$
$$K_2 = 12.7$$ \hspace{1cm} (29)
Analyzing these experimental results, it is clear that trial-to-trial error convergence is possible with acceptable along the trial dynamics, where Fig 7 shows the input (top plot), output (middle plot) and error (bottom plot) on trial 200. For this particular case, the convergence rate is somewhat slow and how to increase this (without sacrificing along the trial performance) is the subject of on-going work. These results do, however, confirm that the ILC design algorithm here can be applied in both simulation and experiment. Also the result of Theorem 2 provides infinitely many solutions for the control law matrices and further research is clearly necessary to provide tools to select the most appropriate set for a given case.

V. CONCLUSIONS

This paper has considered the design of ILC schemes in a 2D linear systems setting and, in particular, the theory of discrete linear repetitive processes. This releases a stability theory for application which demands uniformly bounded along the trial dynamics (whereas previous approaches only demand bounded dynamics over the finite trial length). Here we have shown that this approach leads to a stability condition expressed in terms of an LMI with immediate formulas for computing the control law matrices for the widely encountered case of a single-input single-output discrete linear plant state-space model where the first Markov parameter (or indeed the first $r > 1$) is zero. Many designs have been proposed for this problem in the literature but none of these have been experimentally verified or are able to deal with any other performance specification beyond trial-to-trial error convergence. The results here establish the basic feasibility of this approach in terms of both theory and experimentation. There is a significant degree of flexibility in the resulting design algorithm and current work is undertaking a detailed investigation of how this can be fully exploited.

REFERENCES