Almost Global Convergence on $(d+1)$-Agent Formations Employing Narrow Weak Rigidity in the d-Dimensional Space

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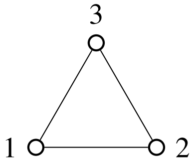
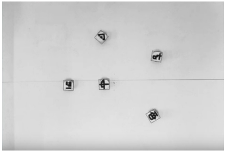
Multi-agent formation

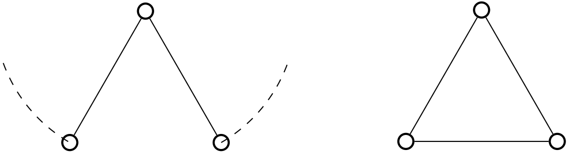
- A geometrical shape formed by multiple agents in a space.
- Represented by a graph (vertices=agents).
- Examples:
  - A group of ground vehicles,
  - A group of flying multi-copters.





Condition for unique formation shape under (distance) rigidity

- (Distance) rigidity is a condition in order that the formation of interest is uniquely defined under given distance constraints.
- (Distance) rigidity has been widely studied in the literature.[1]



(a) Flexible formation. (b) Rigid formation.

[1] L. Asimow and B. Roth. "The rigidity of graphs". In: *Transactions of the American Mathematical Society* 245.11 (1978), pp. 279–289; B. Hendrickson. "Conditions for unique graph realizations". In: *SIAM Journal on Computing* 21.1 (1992), pp. 65–84.

Condition for unique formation shape under narrow weak rigidity

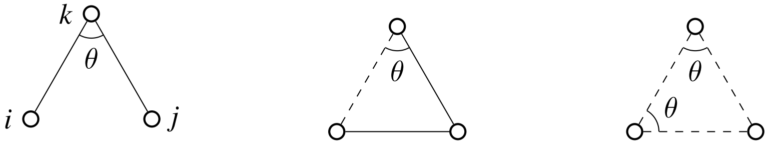
- Rigidity with distance- and subtended angle-constraint is named *narrow weak rigidity*.
- Narrow weak rigidity is a condition in order that the formation of interest is uniquely defined under given distance constraints and subtended angle constraints[2].

(a) Flexible formation. (b) Narrow weak rigid formation.

[2] Myoung-Chul Park, Hong-Kyong Kim, and Hyo-Sung Ahn. "Rigidity of Distance-based Formations with Additional Subtended-angle Constraints". In: *Proc. of the 17th International Conference on Control, Automation and Systems (ICCAS)*.

Comparison of weak rigidity and narrow weak rigidity

- *Weak rigidity*[3] is an extended version of the narrow weak rigidity
- Weak rigidity $\rightarrow$ angle constraint: $\cos\theta_{ij}^{k}$
- Narrow weak rigidity $\rightarrow$ angle constraint: $z_{ik}^{\top} z_{jk} = \|z_{ik}\| \|z_{jk}\| \cos\theta_{ij}^{k}$



(a) Narrow weak rigid formation.

(b) Weak rigid formation.

(c) Weak rigid formation.

[3] Seong-Ho Kwon et al. "Infinitesimal weak rigidity and stability analysis on three-Agent formations". In: *Proc. of the 2018 57th SICE Annual Conference*.

Motivation and contribution

- There is no result on exponential stability of 4-agent formations in 3-dimensional space w.r.t both weak rigidity.
- To control a tetrahedral formation with distance and angle constraints in 3-dimensional space w.r.t narrow weak rigidity.

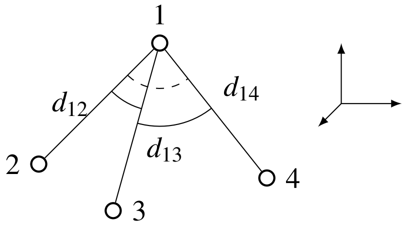


Figure. Tetrahedral formation with 3 distance and 3 angle constraints in $\mathbb{R}^3$.

Notations and terminologies

- For a given undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the vertex set and $\mathcal{E}$ is the edge set.
- $\mathbf{p}_i \in \mathbb{R}^d$: position vector of agent i.
- $\mathbf{p} \triangleq (\mathbf{p}_1, \ldots, \mathbf{p}_n) \in \mathbb{R}^{dn}$ is called a *configuration* in $\mathbb{R}^d$.
- $(\mathcal{G}, \mathbf{p})$ is called a *framework (formation)*.
- d_{ij}: distance constraint assigned to $(i, j) \in \mathcal{E}$.
- θ_{ij}^k: angle constraint assigned to $(k, i, j) \in \mathcal{A}$.
- $\mathbf{z}_{ij} \triangleq \mathbf{p}_i - \mathbf{p}_j$: relative position.
- d(dimension) $= 3$.
- Framework always includes the angle set $\mathcal{A}$.

Infinitesimal narrow weak rigidity (INWR)

- The *narrow weak rigidity function* $F_W^N : \mathbb{R}^{dn} \rightarrow \mathbb{R}^{(m_d+m_a)}$ is defined as
$$F_W^N(\mathbf{p}) \triangleq [\|\mathbf{z}_1\|^2, ..., \|\mathbf{z}_{m_d}\|^2, A_1, ..., A_{m_a}]^\top$$
where $m_d = |\mathcal{E}|, m_a = |\mathcal{A}|$ and $A_h = A_{h_{kij}} \triangleq z_{ik}^\top z_{jk}, (k,i,j) \in \mathcal{A}, h \in \{1, ..., m_a\}$. The rigidity function describes constraints of distances and subtended angles in given framework.
- The *narrow weak rigidity matrix* is defined as the Jacobian of the rigidity function as follows
$$R_W^N(\mathbf{p}) \triangleq \frac{\partial F_W^N(\mathbf{p})}{\partial \mathbf{p}} \in \mathbb{R}^{(m_d+m_a)\times dn}.$$

Infinitesimal narrow weak rigidity (INWR)

- If $\dot{F}_W^N = R_W^N(\mathbf{p})\delta\mathbf{p} = 0$, then $\delta\mathbf{p}$ is called an infinitesimal motion of $(\mathcal{G}, \mathbf{p})$.

Definition (Trivial infinitesimal motion)

An infinitesimal motion of a framework $(\mathcal{G}, p)$ is called *trivial* if it corresponds to rigid-body translation and rotation of the entire framework.

Definition (INWR)

A framework $(\mathcal{G}, p)$ with $\mathcal{A}$ satisfies *infinitesimal narrow weak rigidity* (INWR) in $\mathbb{R}^d$ if all of infinitesimal motions are trivial.

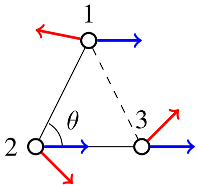


Figure. Trivial infinitesimal motions: rigid-body translation and rotation in $\mathbb{R}^2$.

Infinitesimal narrow weak rigidity (INWR)

Theorem

A framework $(\mathcal{G}, p)$ with $\mathcal{A}$ satisfies infinitesimal narrow weak rigidity (INWR) in $\mathbb{R}^d$ if and only if $R_W^N(p)$ is of rank $dn - d(d+1)/2$[1].

- Rank condition of $R_W^N(p) \Rightarrow$ INWR

[1] Infinitesimal motions correspond to rigid-body translations and rotations of entire framework.

Its application to formation control problem

- We assume that each agent is governed by a single integrator.
- Gradient flow law is employed as follows

$$\dot{p} = u = -(\nabla\phi)^{\top}. \tag{1}$$

where $\phi = \frac{1}{2}e^{\top}(p)e(p)$ and $e(p) = [d_c(p)^{\top} c_c(p)^{\top}]^{\top} - [d_c^{*\top} c_c^{*\top}]^{\top}$.
($d_c(p) = [\ldots, \|z_{g_{ij}}\|^2, \ldots]^{\top}_{(i,j)\in\mathcal{E}}$, $c_c(p) = [\ldots, A_{h_{kij}}, \ldots]^{\top}_{(k,i,j)\in\mathcal{A}}$)

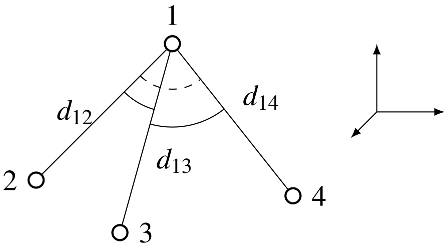


Figure. Tetrahedral formation with 3 distance and 3 angle constraints in $\mathbb{R}^3$.

Its application to formation control problem

$$
\begin{aligned}
\dot{p} = u &= -\left(\nabla\phi\right)^\top \\
&= -{R_W^N}^\top(p)e(p) \\
&= -\begin{bmatrix} 2z_{12}e_1 + 2z_{13}e_2 + 2z_{14}e_3 + \mathbf{e_A} \\ -2z_{12}e_1 - z_{13}e_{A_1} - z_{14}e_{A_2} \\ -2z_{13}e_2 - z_{12}e_{A_1} - z_{14}e_{A_3} \\ -2z_{14}e_3 - z_{12}e_{A_2} - z_{13}e_{A_3} \end{bmatrix}
\end{aligned} \tag{2}
$$

where $\mathbf{e_A} = (z_{12}+z_{13})e_{A_1} + (z_{12}+z_{14})e_{A_2} + (z_{13}+z_{14})e_{A_3}$, $e_1 = \|z_{12}\|^2 - \|z_{12}^*\|^2$, $e_2 = \|z_{13}\|^2 - \|z_{13}^*\|^2$, $e_3 = \|z_{14}\|^2 - \|z_{14}^*\|^2$, $e_{A_1} = z_{12}^\top z_{13} - {z_{12}^*}^\top z_{13}^*$, $e_{A_2} = z_{12}^\top z_{14} - {z_{12}^*}^\top z_{14}^*$, $e_{A_3} = z_{13}^\top z_{14} - {z_{13}^*}^\top z_{14}^*$.

Its application to formation control problem

All equilibria are denoted by

$$\mathcal{P}^* = \{p \in \mathbb{R}^{dn} \mid e = 0\} \text{ :desired equilibrium set,} \quad (3)$$

$$\mathcal{P}_i = \{p \in \mathbb{R}^{dn} \mid {R_W^N}^\top e = 0, e \neq 0\} \text{ :incorrect equilibrium set.} \quad (4)$$

Lemma

In the case of $(d+1)$-agent formations in $\mathbb{R}^d$, an incorrect equilibrium $\bar{p}$ always lies on a hyperplane.

Lemma

Any incorrect equilibrium point $\bar{p}$ of the system (2) is unstable.

Theorem (Main result)

If a framework $(\mathcal{G}, p(0))$ with $\mathcal{A}$ satisfies INWR, then $p(0)$ converges to a point in $\mathcal{P}^*$ as $t \to \infty$ exponentially fast.

Desired formation

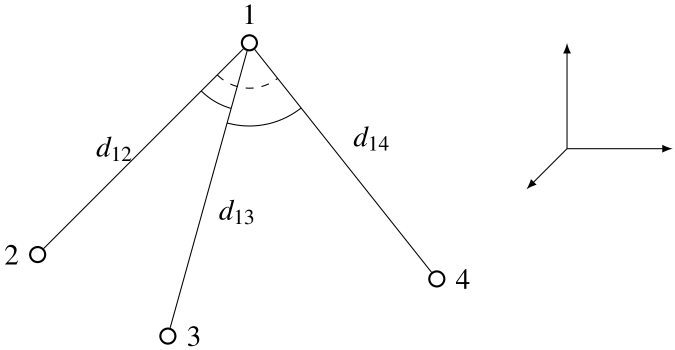
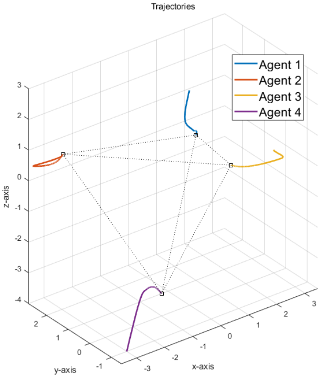
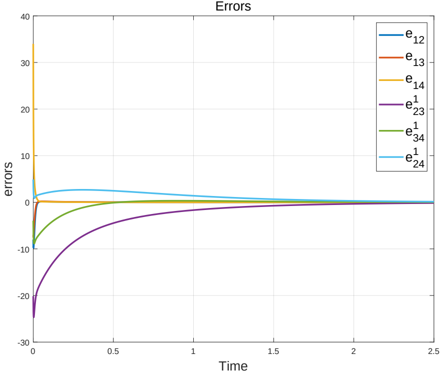


Figure. Tetrahedral formation with 3 distance and 3 angle constraints in $\mathbb{R}^3$.

Simulation



(a) Trajectories of 4-agent formation in $\mathbb{R}^3$.



(b) Errors of 3 distance and 3 angle constraints.

Summary

- Narrow weak rigidity using distance and subtended angle constraints.
- Tetrahedral Formation control in 3-dimensional space with the INWR.
    - Exponential convergence if an initial point satisfies INWR.
- Future work
    - To control a tetrahedral formation in 3-dimensional space with the weak rigidity.

References I

[1] L. Asimow and B. Roth. “The rigidity of graphs”. In: *Transactions of the American Mathematical Society* 245.11 (1978), pp. 279–289.

[2] B. Hendrickson. “Conditions for unique graph realizations”. In: *SIAM Journal on Computing* 21.1 (1992), pp. 65–84.

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[4] Seong-Ho Kwon, Minh Hoang Trinh, Koog-Hwan Oh, Shiyu Zhao, and Hyo-Sung Ahn. “Infinitesimal weak rigidity and stability analysis on three-Agent formations”. In: *Proc. of the 2018 57th SICE Annual Conference*.

Thanks for your attention.