Stability Analysis on Four Agent Tetrahedral Formations

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Formation Control Problems

- Swarm of multi-agent systems[1]
- Rendezvous problem: all members of the group to eventually rendezvous at single unspecified location.[2]
- Formation shape maintenance[3]
- A survey by Oh and his colleagues[4]

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[1] V. Gazi and K. M. Passino. "Stability analysis of swarms". In: *IEEE Transactions on Automatic Control* 48.4 (2003), pp. 692–697.

[2] J. Lin, A. S. Morse, and B. D. O. Anderson. "The multi-agent rendezvous problem". In: *Proceedings of the 42nd IEEE Conference on Decision and Control.* Dec. 2003, pp. 1508–1513.

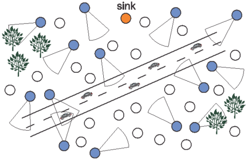
[3] B. D. O. Anderson et al. "Rigid graph control architectures for autonomous formations". In: *IEEE Control Systems Magazine* 28.6 (2008), pp. 48–63.

[4] K.-K. Oh, M.-C. Park, and H.-S. Ahn. "A survey of multi-agent formation control". In: *Automatica (in press)* (2014).

Applications

- To generate a group behavior of multi-agent system (for open purposes)
  - Cooperative surveillance by unmanned aerial vehicles
  - Localization and deployment of wireless sensor networks

(a) Two Parrot AR.Drones (picture by Halftermeyer)



(b) Visual sensor networks (picture by Costa)

Formation Shape Maintenance

- Relative displacements vs. inter-agent distances
- Undirected graph vs. directed graph

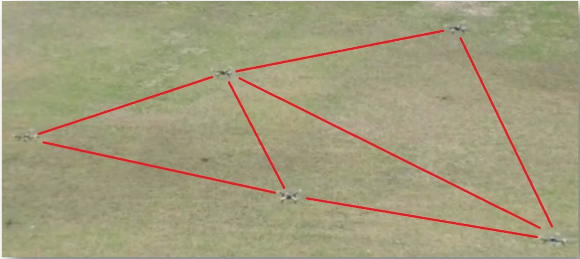


Figure. Formation flying of five quadrotors (picture by Kang)

Problem Formulation

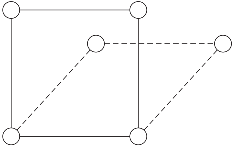
- Objectives
  - To achive a prescribed desired formation shape by maintaining prescribed inter-agent distances only
  - Control input using local measurements (relative displacements) only
- Method
  - Gradient-descent law minimizing a potential function

How to analyze the stability of the desired formation shape?

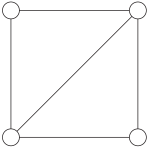
Distance-based Formation Control

- Inter-agent distances are actively controlled.
- The formation graph is required to be rigid.

- The shape of the graph in the first figure cannot be maintained in $\mathbb{R}^2$ by fixed edge lengths unlike the graph in the second figure.



(a) Flexible framework



(b) Rigid framework

$\mathcal{K}_4$ Formation Problem I

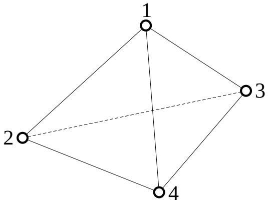
- We consider a group of four mobile agents moving in $\mathbb{R}^3$.
- Each agent is supposed to maintain specified distances from the others.
- Similar problem in 2-dimensional space: Anderson et al. (2010)[5], Dasgupta et al. (2011)[6].

[5] B. D. O. Anderson et al. "Controlling four agent formations". In: *Proceedings of the 2nd IFAC Workshop on Distributed Estimation and Control in Networked Systems*. Sept. 2010, pp. 139–144.

[6] S. Dasgupta et al. "Controlling rectangular formations". In: *Proceedings of the 2011 Australian Control Conference*. Nov. 2011, pp. 44–49.

$\mathcal{K}_4$ Formation Problem II

- Interaction topology $\Leftrightarrow$ Abstract graph $= \mathcal{K}_4$ (complete graph with four vertices) which is rigid in $\mathbb{R}^3$
- Agents $\Leftrightarrow$ Vertices of the graph
- Formation shape $\Leftrightarrow$ Framework (=abstract graph + position vectors of the vertices)



Notation

- Vertex set: $\mathcal{V} = \{1, 2, 3, 4\}$
- Edge set: $\mathcal{E} = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$
- Position vector: $p_i \in \mathbb{R}^3$, $p = [p_1^T \;\; p_2^T \;\; p_3^T \;\; p_4^T]^T \in \mathbb{R}^{3|\mathcal{V}|}$
- Desired formation shape: $\bar{p} \in \mathbb{R}^{3|\mathcal{V}|}$
- Inter-agent distances: $d_{ij} = \|p_i - p_j\|$, $\bar{d}_{ij} = \|\bar{p}_i - \bar{p}_j\|$
- Squared-distance error: $e_{ij} = d_{ij}^2 - \bar{d}_{ij}^2$
    - Objective: $\lim_{t\to\infty} e_{ij} = 0$ for all $(i, j) \in \mathcal{E}$

Assumption I

- Single-integrator dynamics: $\dot{p}_i = u_i$
- Quadratic potential function: $V(p) = \frac{1}{4}\sum_{(i,j)\in\mathcal{E}} e_{ij}^2$
- Gradient-descent law:

$$\dot{p} = -\left[\frac{\partial V}{\partial p}\right]^T = -[R(p)]^T e(p) = -(E(p)\otimes I_3)p.$$

- No mismatched desired distances[7,8]

[7] A. Belabbas et al. "Robustness Issues with Undirected Formations". In: *Proceedings of the 51st IEEE Conference on Decision and Control.* Dec. 2012, pp. 1445–1450.

[8] Z. Sun et al. "Non-robustness of gradient control for 3-D undirected formations with distance mismatch". In: *Proceedings of the 2013 Australian Control Conference.* Nov. 2013, pp. 369–374.

Assumption II

$$R(p) \triangleq \frac{1}{2}\frac{\partial e}{\partial p} = \begin{bmatrix} p_1^T - p_2^T & p_2^T - p_1^T & 0 & 0 \\ p_1^T - p_3^T & 0 & p_3^T - p_1^T & 0 \\ p_1^T - p_4^T & 0 & 0 & p_4^T - p_1^T \\ 0 & p_2^T - p_3^T & p_3^T - p_2^T & 0 \\ 0 & p_2^T - p_4^T & 0 & p_4^T - p_2^T \\ 0 & 0 & p_3^T - p_4^T & p_4^T - p_3^T \end{bmatrix}.$$

- Further we assume that $R(\bar{p})$ has full row rank, which is equivalent[9] that the framework $(\mathcal{K}_4, \bar{p})$ is rigid[10].

[9] B. Hendrickson. "Conditions for unique graph realizations". In: *SIAM Journal on Computing* 21.1 (1992), pp. 65–84.

[10] Rigorously say, infinitesimally rigid

Existing Result

- $p(t)$ approaches equilibrium set as $t \to \infty$.
- The origin of the error dynamics is (locally) exponentially stable.[11]

$$\dot{V} = -eRR^T e \leq -4\left[\lambda_{\text{min}}(RR^T)\right] V \leq 0.$$

  - The matrix RR^T is positive definite near the desired formation shape from the assumption on $\bar{p}$. $\Rightarrow \lambda_{\text{min}}(RR^T) > 0$.

[11] F. Dörfler and B. Francis. "Formation control of autonomous robots based on cooperative behavior". In: *Proceedings of the 2009 European Control Conference*. Aug. 2009, pp. 2432–2437.

Analysis on Incorrect Equilibria I

- Consider incorrect equilibrium set given by

$$\mathcal{P}_i = \left\{ p \in \mathbb{R}^{3|\mathcal{V}|} : \dot{p} = -[R(p)]^T e(p) = -(E(p) \otimes I_3)p = 0,\ e(p) \neq 0 \right\}.$$

- Instability of the incorrect equilibirum set
  - Linearization
  - Existence of positive eigenvalue

Analysis on Incorrect Equilibria II

- Linearization:

$$\frac{\partial}{\partial p}\left[-\frac{\partial V}{\partial p}\right]^T = -H_V(p),$$

  where H_V is the Hessian matrix of V by definition.
- Investigate the existence of negative eigenvalue(s) of H_V.

Analysis on Incorrect Equilibria III

- Let p^* be an element in the incorrect equilibrium set. With an appropriate transformation, we have

$$\bar{H}_V(p^*) = \begin{bmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & E(p^*) \end{bmatrix}.$$

- Show the existence of negative eigenvalu(s) of $E(p^*)$, which can be done by finding a vector x such that $x^T[E(p^*) \otimes I_3]x < 0$.
- Actually, we have

$$x^T[E(p^*) \otimes I_3]x = -\sum_{(i,j)\in\mathcal{E}} [e_{ij}(p^*)]^2 < 0,$$

for $x = \bar{p}$.

Main Result I

Theorem

For almost every initial condition in $\mathbb{R}^{3|\mathcal{V}|}$, the trajectory $p(t)$ of (1) *converges to the desired equilibrium set $\mathcal{P}_d$, where $\mathcal{P}_d = \left\{p \in \mathbb{R}^{3|\mathcal{V}|} : e(p) = 0\right\}$.*

$$\dot{p} = -\left[\frac{\partial V}{\partial p}\right]^T. \tag{1}$$

- What does “almost every initial condition” mean?

Main Result II

- Consider $Z(p) = [r_{12} \;\; r_{13} \;\; r_{14}] \in \mathbb{R}^{3\times3}$, where $r_{ij} = p_i - p_j$. Let $\Delta(p(t)) = \det Z(p(t))$
- $|\Delta|$ is proportional to the volume occupied by the tetrahedron in $\mathbb{R}^3$.
- We can show that $\Delta(p^*) = 0$ for any p^* in the incorrect equilibrium set.

Main Result III

- Suppose that $(\mathcal{K}_4, p^*)$ is a point formation or has a planner shape.
  - We can show that $\Delta(p(t))$ cannot converge to 0 if $p(0)$ is not in $\mathcal{C} = \{p \in \mathbb{R}^{3|\mathcal{V}|} : \Delta(p) = 0\}$.
- Suppose that $(\mathcal{K}_4, p^*)$ is a line formation.
  - We can show that $p(t)$ is able to converge to p^* only if $(\mathcal{K}_4, p)$ is a line formation.

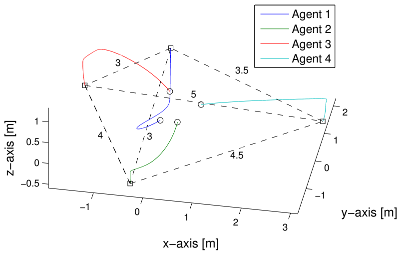
Main Result VI

Corollary

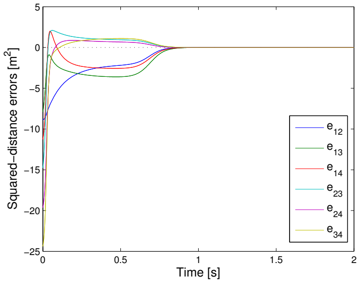
Under (1)*, the region of attraction for the desired equilibrium set* $\mathcal{P}_d$ *is* $\mathbb{R}^{3|\mathcal{V}|} \setminus \mathcal{C}$.

Simulation I

Initial condition: $p_1(0) = [0 \quad 0 \quad 0]^T$, $p_2(0) = [0.5 \quad 0 \quad 0]^T$, $p_3(0) = [0 \;\; 1.5 \;\; 0]^T$, and $p_4(0) = [1 \;\; 1 \;\; 0.001]^T$



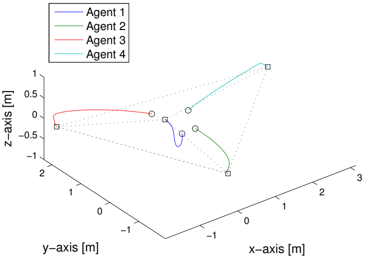
(c) Trajectories of the agents



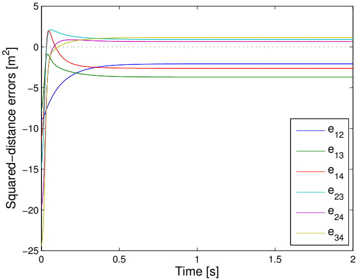
(d) Squared-distance errors

Simulation II

Initial condition: $p_1(0) = [0 \quad 0 \quad 0]^T$, $p_2(0) = [0.5 \quad 0 \quad 0]^T$, $p_3(0) = [0 \ \ 1.5 \ \ 0]^T$, and $p_4(0) = [1 \ \ 1 \ \ 0]^T$



(e) Trajectories of the agents



(f) Squared-distance errors

Conclusion

- Conclusion
  - We provide an analysis on the behavior of $\mathcal{K}_4$ formation with single-integrator modeled agents.
  - Unlike the previous work[12], the desired formation shape is not restricted to be equilateral tetrahedron.
- Future work
  - Consideration of more complex agent dynamics
  - Extention to more general formations having more than four agents
  - Extention to formations having directed graph topology

[12] M.-C. Park, K. Jeong, and H.-S. Ahn. "Control of Undirected Four-agent Formations in 3-dimensional Space". In: *Proceedings of the 52nd IEEE Conference on Decision and Control*. Dec. 2013, pp. 1461–1465.

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