Consensus of Multi-agent Systems with Saturation Constraints

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포화 제약조건들을 고려한 다중 애이전트 시스템의 컨설팅

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Consensus of Multi-agent Systems with Saturation Constraints

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Abstract

In this dissertation, we study the consensus problem for multi-agent systems with saturation constraints. In many real systems, there exist several constraints, such as input saturations, output saturations, or state saturations. Moreover, for large-scale systems, the saturation nonlinearities in interconnections may arise due to the limited flow capacities of the interconnections. Then, under these constraints, the consensus cannot be realized due to the existence of unachievable equilibria for the consensus. While the consensus control problems under such constraints have been addressed in much detail, the analysis problems have received fewer results. Therefore, this dissertation investigates conditions for achieving the consensus under saturation constraints. Specifically, we focus on the analysis problem for two types of saturations as follows:

First, we study the consensus problems under saturation constraints in interconnection states, that are defined by p-norm. By utilizing the edge Laplacian and the state saturation function, the overall system under interconnection constraints are described by the node and constrained edge dynamics. Sufficient conditions are identified under which the given constrained systems achieve the average consensus. Then, as examples of applications, we extend the analytical results to the synthesis problem for achieving rendezvous under limited communication ranges and input constraints, and the balancing problem under flow constraints.

Second, we study the consensus problems under output saturations. We consider single-integrator modeled agents with both fixed and time-varying undirected graphs. Moreover, both homogeneous and heterogeneous saturation levels are considered. For each case, we investigate the attractivity with respect to the equilibria, and some properties of each agent. Then, the necessary and sufficient conditions are derived by investigating the conditions for
equilibrium to be an achievable equilibrium for the consensus. Additionally, we extend the results of single-integrator agent cases to double-integrator agent case as well as fixed and directed graph cases.
국문요약

본 학위 논문에서는 포화 제약조건들을 고려한 다중 에이전트 시스템의 컨센서스(Consensus) 문제를 다루고자 한다. 실제로 많은 시스템들은 입력 포화, 출력 포화, 또는 상태 포화와 같은 다양한 제약조건들을 갖고 있다. 또한, 대규모 시스템들은 제한된 연결상의 수용력 때문에 연결상의 포화 비선형성이 발생할 수도 있다. 이러한 제약조건하에서는 컨센서스에 도달할 수 없는 평형점들이 존재하기 때문에, 컨센서스가 실현되지 않을 수 있다. 비록 이러한 제약조건을 고려한 컨센서스 제어 문제들은 많이 다뤄지고 있는 반면에, 해석에 대한 문제들은 적은 결과만을 찾을 수 있다. 따라서, 본 학위 논문에서는 포화 비선형화하에서 컨센서스가 달성되기 위한 조건들을 조사한다. 구체적으로 다음과 같은 두 종류의 포화 제약조건들에 대하여 해석 문제에 초점을 맞춘다.

첫째, 연결 상태들에 존재하는 성분별, 노름 유계 형태의 포화 제약조건들을 고려한 컨센서스 문제들을 연구한다. 옛지 라플라시안과 상태 포화 함수를 이용하여, 연결 상태의 제약조건하의 전체 시스템은 노드와 고속된 옛지 다투나믹으로 표현된다. 이 구속된 시스템들이 평균 상태 일치를 달성하기 위한 충분 조건들을 알아본다. 응용의 예로써, 분석 결과를 제한된 통신 범위들과 입력 제약조건들을 고려하여 랜덤성을 달성하기 위한 설계 문제와 효율성의 제약이 있는 균형문제로 확장한다.

둘째, 출력 포화를 고려한 일치 문제들을 연구한다. 이 문제에서 고정된 무방향 그래프와 시변하는 무방향 그래프로 고려한 단일 적분기의 에이전트 모델을 고려한다. 또한, 균일한 포화 레벨과 불균일한 포화 레벨 모두를 고려한다. 각 경우에 대하여, 평형점들에 대한 포인트성과 각 에이전트의 특성들에 대하여 조사한다. 이것을 바탕으로, 평형점이 컨센서스를 달성할 수 있는 평형점이 되기 위한 조건들을 조사함으로써 필요 충분 조건들을 규정한다. 추가적으로, 단일 적분기 에이전트의 결과를 이중 적분기 에이전트와 고정된 방향 그래프로 확장한다.
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Chapter 1

Introduction

1.1 Background

This thesis is concerned with the study of consensus problem for multi-agent systems. The multi-agent system consists of a number of multiple interacting agents. Many engineering systems can be modeled as multi-agent systems such as electric power grids, communication networks, biological systems, etc.

Distributed control for the multi-agent systems has been widely studied over the last decades due to a number of benefits, such as increased robustness, efficiency and flexibility, etc [1]. In the distributed control problem, the main objective is to enable a group of agents to perform tasks using local information and/or resource sharing.

In the study of distributed control for multi-agent systems, consensus problems have attracted by many researchers due to their wide applications such as flocking behavior analysis, sensor network, unmanned air vehicle (UAV) formations, rendezvous, etc (see, e.g., [2–5]). The consensus means to reach an agreement about common behavior in a group, and the consensus algorithm is an interaction rule that specifies the information exchange between agents.

The interaction between agents can be modeled by a graph. Hence, a graph theory plays an important role in the consensus problem. Specifically, graph laplacian, which describes
the connectivity of the network, and their spectral properties [6, 7] are used to analyze the convergence of consensus. Then, utilizing the properties of the graph laplacian, the theoretical framework of the consensus problems for single-integrator modeled agents was studied in [8-11]. These results have been extended to double-integrator agents [12-14] and general linear systems [15-17]. In [18], the result in [15] was extended to the robust consensus problem for agents with uncertainty in the form of additive perturbations. Moreover, the local and global consensus conditions and the distributed adaptive consensus algorithms have been investigated for Lipschitz nonlinear systems in [19] and [20], respectively.

On the other hands, many real systems are subject to physical constraints such as input, output and communication constraints, etc. Due to the existence of interactions among agents, the constraints appear over the network, and have an effect on the behaviors of the agents. As a result, the consensus is dependent on not only the graph topology and the system dynamics, but also the constraints. Therefore, the goal of this thesis is to analyze the effect of the constraints on the consensus problem.

1.2 Overview of Related Work

In this chapter, we summarize the related work on the consensus problem under several constraints. Table. 1.1 provides a brief summary and comparison.

1.2.1 State Constraints

In practice, the states of each agent have constraints due to the physical limitations and/or the safety. Therefore, in [21], a constrained consensus algorithm based on a nonlinear projec-
tor was proposed to solve the consensus problem under state constraints such that each agent stayed in its own convex set, and convergence under a dynamically changing balanced graph was analyzed. An extension of [21] to an unbalanced graph with communication delays was studied in [22]. Moreover, in many applications such as distributed averaging(balancing) problem for storage devices, each agent is controlled by flows between agents. Therefore, the constrained consensus problems using flow controls have been studied in [23][24]. [23] proposed a non-iterative, distributed algorithm under state and flow constraints. In [24], the constrained consensus algorithm proposed in [21] was extended to the flow control problem.

To deal with the state constraints, a containment control problem have been studied in [25][28]. The containment control is to drive the states of the agents into the convex hull spanned by given points, called leaders, and usually formulated as a tracking problem. In [25], a hybrid Stop-Go algorithm was proposed to solve this problem for the stationary/moving leaders under the fixed undirected graph. In [26], nonlinear containment algorithms was developed to deal with the switching directed graph. Recently, these results was extended to double-integrator agents [27] and general linear systems [28].

Meanwhile, for multi-agent systems, the constraints in the interconnection states may arise when the interconnections represents the flows of certain amount resources since the flows used for the resource transfers have the limited capacity. Hence, in recent years, the consensus problem under flow constraints have been studied in [23][29]. Moreover, in the case of the communication networks, each agent interacts using the communication modules. Due to the existence of the limited communication ranges, the interconnections have the communication range constraints. The communication range constraints cause the loss of
connectivities. Therefore, the connectivity preserving algorithms have been developed in [30–32].

Although the state constraints have been studied for general linear systems, e.g., [28], the above results, that deal with the interconnection constraints, have focused on the single-integrator agents. Therefore, this thesis studies the consensus problem for general linear systems under the interconnection constraints.

### 1.2.2 Saturation Constraints

Real engineering systems are subject to saturation constraints. Most common saturation constraint is on amplitude of an actuator. In this case, the saturation constraints appears over the overall network, and consequently, it is difficult to utilize the properties of the Laplacian matrix. Furthermore, it is well known that global asymptotic stabilization under the input saturations can be achieved only when the system is asymptotic null controllable with bounded controls (ANCBC) [33]. Therefore, the consensus problem under input saturations has been studied for ANCBC systems. The consensus problem for integrator systems under input saturations has been studied in [13][14][34]. Recently, these results have been extended to high-order systems whose dynamics is ANCBC [35][36]. By utilizing the low-gain approach, the semi-global consensus problem was studied in [35]. For a group of neutrally stable systems and double-integrator agents, the global consensus problem was studied in [36].

Another example of the saturation constraints is the output saturations due to the limited capacity of a sensor. In the presence of the output saturations, there exists unachievable equilibrium points for the consensus (see, Chapter 4 for details). Therefore, under the output
Table 1.1: Comparison of consensus under constraints ($x_i$: state, $u_i$: input, $y_i$: output)

<table>
<thead>
<tr>
<th>Constraints</th>
<th>State constraints (Containment control)</th>
<th>Interconnection constraints</th>
<th>Input saturation</th>
<th>Output saturation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analysis</td>
<td>$x_i \in \text{co}{x_{n+1}, x_{n+2}, \ldots}$</td>
<td>$|x_i - x_j|_p \leq \Delta$</td>
<td>$|u_i|_p \leq \Delta$</td>
<td>$|y_i|_p \leq \Delta$</td>
</tr>
<tr>
<td>Synthesis</td>
<td>$[21]$</td>
<td>$[23]$</td>
<td>$[36]$</td>
<td>$[38]$</td>
</tr>
</tbody>
</table>

saturations, the synthesis problems for achieving the consensus have been studied in $[37] [38]$.

Although the consensus problem under the input saturations has been studied, the output saturation has rarely been addressed. Moreover, the above results, dealt with the output saturations, have focused on the synthesis problem. Therefore, this thesis mainly focuses on the consensus analysis under the output saturations.

1.3 Contributions and Outline of the thesis

In this thesis, we deal with two types of the saturation constraints: 1) the saturation constraints in the interconnection states; 2) the output saturation constraints. Then, the goal is to find conditions for achieving the consensus. Specifically, the organization and the main contributions from each chapter are described as follows:

In Chapter 2 we briefly present a review of basic concepts of algebraic graph theory. Notions of connectivity for fixed and for time-varying graphs, definitions and some properties of graph Laplacian and edge Laplacian are presented. Then, based on the algebraic graph theory, an overview of basic consensus theory is provided.
In Chapter 3, we focus on consensus algorithms with saturation nonlinearities in the interconnection states. We consider more general system model and more general constraints compared to the existing results, and new analysis methods using the state saturation functions are presented. Then, we provide two examples of applications. First, we construct a consensus algorithm based on the proposed method to solve a rendezvous problem under limited communication ranges and input constraints. Compared with the existing results that require the product of the gradients of the potential functions or the bounded and continuous nonlinear functions, the proposed algorithm is simple, and needs less computation. Furthermore, the proposed algorithm also guarantees consensus with any input constraints by adjusting the control gain determined by the communication range constraint and the number of neighbors. Second, a load balancing problem under flow constraints is analyzed by the proposed method.

In Chapter 4, we address the problem of output saturations in consensus algorithms. Due to the existence of undesired equilibria under output saturations, the consensus may not be realized in a global sense. Therefore, we investigate the conditions for achieving the consensus. We first consider single-integrator modeled agents. Moreover, both fixed and time-varying graphs, and both homogeneous and heterogeneous saturation levels are considered. For each case, we investigate the attractivity with respect to the equilibria, and some behavior properties of each agent. Then, we provide the necessity and sufficient condition for achieving the consensus under output saturations. Next, we extend the results of single-integrator agent cases to double-integrator agent case as well as fixed and directed graph cases. Since the analysis technique in this problem rely on the strictly increasing property of the saturation
within its bounds, we can easily extend the results to the general nonlinearities, which are strictly increasing within its bounds.

In Chapter 5, we conclude this dissertation discussing future works.
Chapter 2

Preliminaries

In this chapter, we provide basic concepts and notions, that will be used in this dissertation. We first summarize basic graph notions and associated algebraic graph theory. Then, we briefly provide an overview of basic consensus theory, which is addressed in [2,4,39].

2.1 Graph Theory

Graph theory is widely used to understand and solve many real-world problems. In this chapter, we briefly review some basic concepts from graph theory [7,40].

2.1.1 Definitions and Notations of Graph

A (fixed) graph $G$ is defined as three-tuple $(V, E, A)$, where $V$ denotes the set of nodes (or vertices), $E \subseteq V \times V$ denotes the set of edges (or arcs), and $A = [\alpha_{ij}] \in \mathbb{R}^{N \times N}$ denotes the underlying weighted adjacency matrix defined as

$$\alpha_{ij} := \begin{cases} \alpha_{ij}, & (i, j) \in E, \\ 0, & (i, j) \notin E, \end{cases}$$

where $\alpha_{ij}$ is the weight assigned to edge $(i, j)$.

We say that a graph $G = (V, E, A)$ is undirected if $(i, j) \in E$, then $(j, i) \in E$. Thus, $\alpha_{ij} = \alpha_{ji}, \forall i, j \in V$, and the adjacency matrix of the undirected graph is symmetric, i.e.,
\( A^T = A \). In this dissertation, self-edges are not allowed, i.e., \((i, i) \notin \mathcal{E}\), and \(\alpha_{ii} = 0 \ \forall i \in \mathcal{V}\). We say that a node \( j \) is a neighbor of node \( i \) if \((i, j) \in \mathcal{E}\). The set of neighbors of a node \( i \) is \( \mathcal{N}_i \).

If a direction is assigned to the edges, the graph is called a directed graph. For a directed edge \((i, j) \in \mathcal{E}\), \( j \) is called head and \( i \) is called the tail of edge \((i, j)\). When every weight of an weighted graph is identical, i.e., \(\alpha_{ij} = 1 \) if \((i, j) \in \mathcal{E}\), and \(\alpha_{ij} = 0 \) otherwise, the graph \( \mathcal{G} \) is said to be an unweighted, and often denoted by \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \). A graph \( \mathcal{G} \) is said to be time-varying if it changes over time \( t \), and denoted by \( \mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), \mathcal{A}(t)) \).

2.1.2 Connectedness of Graphs

A directed path is a sequence of edges of the form \((i_1, i_2), (i_2, i_3), \ldots \). Node \( i \) is said to be connected to node \( j \) if there is a directed path from \( i \) to \( j \). A directed cycle is a directed path that starts and ends at the same node. The distance between two nodes \( i \) and \( j \) is the length of the shortest path between them. Then, the connectivity of undirected and directed graphs can be defined as follows:

**Definition 2.1.1 (Connectivity of undirected Graph)** An undirected graph \( \mathcal{G} \) is called connected if there exists a path between any two distinct nodes.

**Definition 2.1.2 (Connectivity of directed Graph)**

- A directed graph \( \mathcal{G} \) is called strongly connected if there exists a directed path between any two distinct nodes.
A directed graph $\mathcal{G}$ is called weakly connected if there exists an undirected path between any two distinct nodes.

We next summarize the connectivity of an undirected, time-varying graph $\mathcal{G}(t)$ in the followings. For a directed graph, the connectivity can be directed defined from the following definitions.

For a time-varying graph $\mathcal{G}(t)$, we consider the following integral graph:

**Definition 2.1.3** ([39]) (Integral Graph) Given a time-varying graph $\mathcal{G}(t) = (V, \mathcal{E}(t), \mathcal{A}(t))$, the integral graph of $\mathcal{G}(t)$ on $[0, \infty)$ is a constant graph $\bar{\mathcal{G}}_{[0, \infty)} := (V, \bar{\mathcal{E}}, \bar{\mathcal{A}})$, where $V$ is the same node set of $\mathcal{G}(t)$, and $\bar{\mathcal{A}} = [\bar{\alpha}_{ij}] \in \mathbb{R}^{N \times N}$ is defined by

$$
\bar{\alpha}_{ij} = \begin{cases} 
1, & \text{if } \int_{0}^{\infty} \alpha_{ij}(t)dt = \infty, \\
0, & \text{if } \int_{0}^{\infty} \alpha_{ij}(t)dt < \infty.
\end{cases}
$$

**Definition 2.1.4** ([39]) (Integral connectivity of time-varying graph) A time-varying graph $\mathcal{G}(t)$ is said to be integrally connected over $[0, \infty)$ if its integral graph $\bar{\mathcal{G}}_{[0, \infty)}$ is connected.

**Remark 2.1.1** ([39]) If an undirected time-varying graph $\mathcal{G}(t)$ is integrally connected over $[0, \infty)$, then there exists a time interval $0 = t_0 < t_1 < \cdots < t_k < \cdots$ such that $\int_{t_{k-1}}^{t_k} L(t)dt$ is connected $\forall k \geq 0$.

Analogous criterion referred as the “$\delta$-connected graph” was studied in [41]. An edge $(i, j)$ is said to be a $\delta$-edge of $\mathcal{G}(t)$ on time interval $[t_{k-1}, t_k)$ if $\int_{t_{k-1}}^{t_k} \alpha_{ij}(t) \geq \delta$. Then, a time-varying graph $\mathcal{G}(t)$ is said to be uniformly $\delta$-connected if there exists a constant $T > 0$ such that for any $t \geq 0$, the $\delta$-edges of $\mathcal{G}(t)$ on time interval $[t, t + T)$ form a connected
graph. If for any \( t \geq 0 \), the \( \delta \)-edges of \( \mathcal{G}(t) \) on time interval \( [t, \infty) \) form a connected graph, then the graph \( \mathcal{G}(t) \) is said to be infinitely \( \delta \)-connected.

### 2.1.3 Graph Laplacian and Edge Laplacian

The Laplacian matrix (or connectivity matrix) is used to describe the connectivity in a more compact form.

**Definition 2.1.5 (Graph Laplacian)**

Given a graph \( \mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), \mathcal{A}(t)) \), the matrix

\[
L(t) := \mathcal{D}(t) - \mathcal{A}(t)
\]

is called the (graph) Laplacian of \( \mathcal{G}(t) \), where \( \mathcal{D}(t) := \text{diag}(\mathcal{A}(t)1_N) \in \mathbb{R}^N \) and \( 1_N \) is the \( N \times 1 \) column vector comprising all ones.

From **Definition 2.1.5**, we know that the Laplacian matrix \( L \) is diagonally dominant and has nonnegative diagonal entries. Moreover, \( L \) has zero row sums, and thus, zero is an eigenvalue of \( L \) with an associated eigenvector \( 1 \). We can summarize the properties of \( L \) as follows:

**Lemma 2.1.1 (Properties of graph Laplacian matrix : directed graph)**

Given a directed graph \( \mathcal{G} \),

- the corresponding Laplacian matrix \( L \) has at least one zero eigenvalue with \( 1 \) as a corresponding right eigenvector and all nonzero eigenvalues have positive real parts.

- zero is a simple eigenvalue of \( L \) if and only if \( \mathcal{G} \) has a directed spanning tree.
• there exists a nonnegative left eigenvector \( r \) of \( L \) associated with the zero eigenvalue, satisfying \( r^T L = 0 \) and \( r^T 1 = 1 \). Moreover, \( r \) is unique if \( G \) has a directed spanning tree.

**Lemma 2.1.2** (Properties of graph Laplacian matrix : undirected graph) Given an undirected graph \( G \), zero is a simple eigenvalue of \( L \) if and only if the graph is connected, and the remaining \( N - 1 \) eigenvalues are ordered in increasing order as \( 0 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N \).

We next consider an undirected graph \( G = (V, E) \) containing \( N \) nodes and \( M \) edges. An orientation of the undirected graph \( G \) is the assignment of directions to its edges, i.e., an ordered pair \( (i, j) \in E \) such that \( i \) and \( j \) are, respectively, the head and the tail nodes. Then, the incidence matrix \( D = [d_{ij}] \in \mathbb{R}^{N \times M} \) for an oriented graph \( G \) is a \( \{0, \pm 1\} \) matrix, with \( d_{ij} = 1 \) if the node \( i \) is the head of the edge \( j \), \( d_{ij} = -1 \) if it is the tail of the edge \( j \), and \( d_{ij} = 0 \) otherwise. Then, the graph Laplacian can be defined using the incidence matrix \( D \) as follows:

\[
L = DD^T. \tag{2.2}
\]

Moreover, using the incidence matrix \( D \), the edge Laplacian can be defined as follows:

**Definition 2.1.6** [5] (Edge Laplacian) Given an undirected graph \( G = (V, E) \), the matrix

\[
F := D^T D \tag{2.3}
\]

is called the edge Laplacian of an arbitrary oriented graph \( G \), where \( D \in \mathbb{R}^{N \times M} \) is the incidence matrix of an arbitrary oriented graph \( G \).

The edge Laplacian is a real symmetric matrix, and the non-zero eigenvalues of \( F \) are equal to the non-zero eigenvalues of \( L \).
2.2 Consensus Algorithm

**Definition 2.2.1** *(Basic concepts of consensus)* Consider a network of agents

- **Consensus**: Reaching an agreement regarding a certain value of interest.
- **Consensus algorithms**: Iterative schemes that specifies the information exchange among agents to reach consensus.
- **Interaction topology**: Graphs which represent the information state exchange between agents.

In a continuous-time implementation, a simple algorithm to reach an agreement, which is called a standard consensus algorithm, can be expressed as follows:

\[
\dot{x}_i(t) = \sum_{j=1}^{N} \alpha_{ij}(t)(x_j(t) - x_i(t)), \quad (2.4)
\]

where \(\alpha_{ij}(t)\) is the \(ij\)-th entry of the adjacency matrix associated with the underlying graph \(G(t) = (V, E(t), A(t))\). Let \(x(t) = [x_1(t), ..., x_N(t)]^T\), then the overall network can be written as

\[
\dot{x}(t) = -L(t)x(t), \quad (2.5)
\]

where \(L(t) \in \mathbb{R}^{N \times N}\) is the Laplacian matrix associated to the graph.

**Definition 2.2.2** The consensus of \(N\) agents *(2.4)* is said to be achieved if, for all \(i, j = 1, ..., N\), we have 
\[
\lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0.
\]

Assume that the interaction topology is undirected. Then, an equilibrium of the system *(2.5)* is a state of the form \(x^* = C1\), with \(C \in \mathbb{R}\), and the solutions to the consensus problem are given in the following theorems:
Theorem 2.2.1  \[2\] (Fixed graph) Let $G = (V, E, A)$ be a fixed, undirected graph. Then, the system (2.5) is reached consensus for all initial conditions $x(t_0) \in \mathbb{R}^N$ if and only if the graph $G$ is connected. Moreover, the consensus value is given by the average of the initial conditions, i.e., $\lim_{t \to \infty} ||x(t) - \mathbf{1}_N \frac{1}{N}^T x(0)\mathbf{1}_N|| = 0$.

Theorem 2.2.2  \[39\] (Time-varying graph) Let $G(t) = (V, E(t), A(t))$ be a time-varying, undirected graph. Then, the system (2.5) is reached consensus for all initial conditions $x(t_0) \in \mathbb{R}^N$ if and only if the graph $G(t)$ is integrally connected over $[0, \infty)$. Moreover, the consensus value is given by the average of the initial conditions, i.e., $\lim_{t \to \infty} ||x(t) - \mathbf{1}_N \frac{1}{N}^T x(0)\mathbf{1}_N|| = 0$. 
Chapter 3

Consensus with saturation constraints in interconnection states

3.1 Introduction

In practice, there exist several constraints in real systems. It is well known that the dynamical systems with saturation nonlinearities often arise. The saturation nonlinearities can be usually categorized as actuator, sensor or state saturations \[42-45\]. On the other hand, for large-scale systems, the constraints in the interconnections may arise. For the large-scale systems such as production-distribution systems, power networks, transportation networks, etc, the interconnections represent the flows of a certain amount of resources. In this case, there exist the interconnection constraints since the interconnections used for the resource transfers have the limited capacity. Therefore, in recent years, some papers have dealt with the interconnection constraints in the consensus problem. In \[23\], the average consensus problem for networked single-integrators under flow constraints which is associated with the balancing problem for networked storage devices was studied. In \[37\], the discarded consensus algorithm, that discards the state of neighbors if the neighbors are outside its constraints, was developed. In \[29\], the power distribution using consensus-based control was studied under energy flow constraints. In the above papers, the flows are constrained
within the componentwise bounds.

Another example is the cyber interconnected systems such as sensor (or communication) networks. In this case, the interconnections represent the exchange of the informations with neighbors. Then, the interconnections are constrained due to the limited sensing (or communication) ranges, represented by the distance (norm) constraints. In [30–32], the consensus problem with connectivity preservation was studied using the potential-based consensus algorithms. The algorithm proposed in [30] used the unbounded potential field, where unboundedness occurs when the agents approach to the communication limit. However, the large control inputs are not realizable in practice. Hence, in [31], the bounded connectivity preserving algorithm was proposed. They showed that the proposed algorithm remains bounded according to the initial states and energy, but it is not clear whether the proposed algorithm can guarantee the control input bounds or not. In [32], the novel connectivity preserving algorithm whose control input is norm bounded was proposed.

This chapter studies the consensus analysis and synthesis problem under interconnection constraints, that are defined by p-norm. Using the incidence matrix and the edge Laplacian [5], the overall dynamics is governed by the node and edge dynamics. Since the edge dynamics describes the interconnection network behavior, the edge dynamics under the interconnection constraints can be modeled with the state saturations [44][45]. The main contributions of this chapter are as follows: (1) We provide sufficient conditions for achieving consensus under interconnection constraints. Compared to the existing consensus results mentioned above, we consider more general conditions, namely with general linear systems, rather than single-integrators, and with more general constraints, p-norm bounded constraints. (2) We
extend the stability problems for single systems under state saturations [44, 45] to the consensus problem for large scale systems under saturation constraints in interconnection states. Based on Lyapunov approach and LaSalle’s invariance theorem, we derive the asymptotic convergence to the centroid. (3) As an example of applications, we show that the analysis technique can be used for rendezvous problem under limited communication ranges and input constraints. The proposed algorithm forces the edge to stay within its limit by computing the consensus algorithm only in the boundary of the communication range. Compared with the existing results that require the product of the gradients of the potential functions [31] or of the bounded and continuous nonlinear functions [32], the proposed algorithm is simple, and needs less computation. Furthermore, the proposed algorithm also guarantees consensus with any input constraints, and the control gain is determined by the communication range constraint and the number of neighbors. Moreover, balancing problem for storage devices under flow constraints are analyzed. We show that the result in [46] can be analyzed using the proposed method.

3.2 Problem statement

Consider a multi-agent system consisting $N$ agents with diffusive coupling, which is described by

$$\dot{x}_i = Ax_i + \sum_{j=1}^{N} \alpha_{ij} \Gamma(x_j - x_i), \quad i = 1, ..., N, \quad (3.1)$$

where $x_i \in \mathbb{R}^n$ is the state of agent $i$, $\Gamma = \text{diag}(\gamma_1, ..., \gamma_n) \in \mathbb{R}^{n \times n}$ is the inner coupling matrix, $\alpha_{ij} \in \{0, 1\}$ represents the interconnection topology of the network, and $A \in \mathbb{R}^{n \times n}$ is a constant system matrix.
In this chapter, we consider each agent as a node and the interconnection between two agents as an undirected path in an undirected graph. Furthermore, we assume that the interconnection topology is fixed. Thus, the interconnection topology can be described by an (unweighted) undirected graph $G = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{1, ..., N\}$.

In the interconnection graph $G$, we assume that there are $M$ edges, which represent the interconnections, and define the $k$-th edge state $z_k \in \mathbb{R}^n$ as

$$z_k = \sum_{l=1}^{N} d_{lk}x_l, \quad k = 1, ..., M,$$

where $d_{lk}$ is the $lk$th entry of the incidence matrix $D \in \mathbb{R}^{N \times M}$ defined in Chapter 2.1.3. Then, dynamics of agent $i$ can be rewritten as

$$\dot{x}_i = Ax_i - \sum_{k=1}^{M} d_{ik}\Gamma z_k$$

and

$$z_k = \sum_{l=1}^{N} d_{lk}x_l.$$

Let $x = [x_1^T, ..., x_N^T]^T$ and $z = [z_1^T, ..., z_M^T]^T = (D^T \otimes I)x$. Then, the overall dynamics can be written in terms of node and edge dynamics as follows:

$$\dot{x} = (I \otimes A)x - (D \otimes \Gamma)z$$

$$\dot{z} = (I \otimes A - F \otimes \Gamma) z,$$

where $F = [f_{ij}] \in \mathbb{R}^{M \times M}$ is the edge Laplacian matrix (see, Definition 2.1.6).

We next consider nonempty closed convex sets $\Omega_k \in \mathbb{R}^n$, for $k = 1, ..., M$, and will be further specified later. Each set $\Omega_k$ represents the constraint on the edge $k$. Then, the goal of this chapter is to study the consensus analysis and synthesis problem, i.e., $\lim_{t \to \infty} ||x_i(t) - x_j(t)|| = 0$, $\forall i, j \in \mathcal{V}$, under the following two types of interconnection constraints: (a)
componentwise constraints (i.e., $\Omega_k$ is a hypercube), (b) $p$-norm bounded constraints (i.e., $\Omega_k$ is a Euclidean ball).

### 3.3 Analysis under componentwise interconnection constraints

In this chapter, we assume that the constrained sets $\Omega_k$ for $k = 1, \ldots, M$ are defined by the componentwise constraints as follows:

$$\Omega_k := \{ z_k \in \mathbb{R}^n : -\Delta \leq z_{m,k} \leq \Delta, \quad \Delta > 0, \ m = 1, \ldots, n \}. \quad (3.3)$$

Let the boundary of the set (3.3) as $\partial \Omega_k := \bigcup_{m=1}^n \partial \Omega_{m,k}$, where $\partial \Omega_{m,k} = \{ z_k \in \mathbb{R}^n : |z_{m,k}| = \Delta, |z_{l,k}| \leq \Delta, l = 1, \ldots, m - 1, m + 1, \ldots, n \}$. Then, under the interconnection constraints (3.3), we can rewrite the agent $i$ as follows:

$$\dot{x}_i = Ax_i - \sum_{k=1}^M d_{ik} \Gamma z_k$$

$$\dot{z}_k = h(\xi_k),$$

$$\xi_k = Az_k - \sum_{l=1}^M f_{kl} \Gamma z_l,$$

where $z_k \in \Omega_k \in \mathbb{R}^n$ for $k = 1, \ldots, M$, and $h(\cdot)$ represents the interconnection constraints defined by $h(\xi_k) = [h(\xi_{1,k}), \ldots, h(\xi_{n,k})]^T$, with, for $m = 1, \ldots, n$,

$$\dot{z}_{m,k} = h(\xi_{m,k}) := \begin{cases} 0, & \text{if } z_k \in \partial \Omega_{m,k} \text{ and } \xi_{m,k} \cdot z_{m,k} \geq 0, \\ \xi_{m,k}, & \text{otherwise}. \end{cases} \quad (3.5)$$

Let $x = [x_1^T, \ldots, x_N^T]^T$, $z = [z_1^T, \ldots, z_M^T]^T$, and $\xi = [\xi_1^T, \ldots, \xi_M^T]^T$. Then, the overall
dynamics of (3.4) can be expressed in a compact form as

\[ \dot{x} = (I \otimes A)x - (D \otimes \Gamma)z \]

\[ \dot{z} = h(\xi) \]

\[ \xi = (I \otimes A - F \otimes \Gamma)z. \]

The function \( h(\cdot) \) is called a state saturation \([44]\) that is used to describe the saturation constraints in the state variables. From the definition of the saturation function \( h(\cdot) \) (3.5), it is clear that the constrained sets \( \Omega_k \) for \( k = 1, ..., M \) are invariant, i.e., if \( z_k(0) \in \Omega_k \), then \( z_k(t) \in \Omega_k \) for all \( t \geq 0 \). Hence, the saturation function \( h(\cdot) \) forces the states to stay within its constrained set. To handle the saturation function \( h(\cdot) \), we consider the following lemma modified from \([44]\).

**Lemma 3.3.1** Given (3.4), consider a symmetric positive definite matrix \( P = [p_{ij}] \in \mathbb{R}^{n \times n} \) satisfying \( p_{ii} \geq \sum_{j=1, j \neq i}^{n} |p_{ij}|, \ i = 1, ..., n \). Then,

\[ z^T (I \otimes P) h(\xi) \leq z^T (I \otimes P) \xi. \] (3.6)

**Proof:** To prove the inequality (3.6), we first subtract the left-hand side from the right as follows:

\[ z^T (I \otimes P) (\xi - h(\xi)) = \sum_{i=1}^{n} \sum_{k=1}^{M} P_{i,k} (\xi_{i,k} - h(\xi_{i,k})) \]

\[ = \sum_{i=1}^{n} \sum_{k=1}^{M} \left( p_{ii} z_{i,k} + \sum_{j=1, j \neq i}^{n} p_{ij} z_{j,k} \right) (\xi_{i,k} - h(\xi_{i,k})). \] (3.7)

We next define the finite index sets \( K := \{1, ..., M\}, I := \{1, ..., n\}, \) and \( S \) representing the saturated components as:

\[ S := \{(k, i) \in K \times I : z_k \in \partial \Omega_{i,k} \text{ and } \xi_{i,k} \cdot z_{i,k} > 0\}, \]
and the complement of $\mathcal{S}$ as $\mathcal{S}^c := \{(k, i) \in \mathcal{K} \times \mathcal{I} : (k, i) \notin \mathcal{S}\}$ representing the unsaturated components. Then, from (3.5), $h(\xi_{i,k}) = \xi_{i,k}$ if $(k, i) \in \mathcal{S}^c$, and $h(\xi_{i,k}) = 0$ if $(k, i) \in \mathcal{S}$, with $|z_{i,k}| = \Delta$ for $(k, i) \in \mathcal{S}$. Hence, (3.7) is equivalent to

$$
\sum_{i=1}^{n} \sum_{k=1}^{M} \left( p_{ii}z_{i,k} + \sum_{j=1, j \neq i}^{n} p_{ij}z_{j,k} \right) \left( \xi_{i,k} - h(\xi_{i,k}) \right) = \sum_{(k,i) \in \mathcal{S}} \left( p_{ii}z_{i,k} + \sum_{j=1, j \neq i}^{n} p_{ij}z_{j,k} \right) \xi_{i,k} - \sum_{(k,i) \in \mathcal{S}} \left( p_{ii} + \sum_{j=1, j \neq i}^{n} p_{ij} \frac{z_{j,k}}{z_{i,k}} \right) z_{i,k} \cdot \xi_{i,k}. \tag{3.8}
$$

Since, for $(k, i) \in \mathcal{S}$, $\left| \frac{z_{j,k}}{z_{i,k}} \right| \leq 1$, $j = 1, \ldots, n$, $j \neq i$, we have from the condition on $P$ that

$$
\left( p_{ii} + \sum_{j=1, j \neq i}^{n} p_{ij} \frac{z_{j,k}}{z_{i,k}} \right) \geq 0.
$$

Moreover, for $(k, i) \in \mathcal{S}$, $z_{i,k} \cdot \xi_{i,k} > 0$. Therefore, we can conclude that (3.8) $\geq 0$, which implies the condition (3.6).

Then, we can analyze the consensus of constrained dynamics (3.4) as follows:

**Theorem 3.3.1** Consider an undirected and connected graph $\mathcal{G}$. The group of $N$ agents (3.1) under the componentwise constraints (3.3) achieves the consensus for all $z_k(0) \in \Omega_k$, $k = 1, \ldots, M$, if there exists a symmetric positive definite matrix $P = [p_{ij}] \in \mathbb{R}^{n \times n}$ satisfying

$$
p_{ii} \geq \sum_{j=1, j \neq i}^{n} |p_{ij}|, \quad i = 1, \ldots, n, \tag{3.9}
$$

such that

$$
(A - \lambda_i \Gamma)^T P + P (A - \lambda_i \Gamma) < 0, \quad i = 2, \ldots, N, \tag{3.10}
$$

where $\lambda_i$ are the eigenvalues of the Laplacian matrix $L$. 

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Proof: Consider a Lyapunov function candidate \( V = x^T (D \otimes I) (I \otimes P) (D^T \otimes I) x = z^T (I \otimes P) z \) with \( P \) satisfying (3.9). Note that \( V \geq 0 \) with \( V = 0 \) only when \( z = (D^T \otimes I) x = 0 \), that is \( x \in \text{span}\{1\} \). Then, the time derivative of \( V \) is given by

\[
\dot{V} = 2z^T (I \otimes P) h (\xi) = 2z^T (I \otimes P) \xi - 2z^T (I \otimes P) (\xi - h (\xi)).
\]

According to Lemma 3.3.1, we know that \( z^T (I \otimes P) (\xi - h (\xi)) \geq 0 \), and, using the definitions of the graph Laplacian and the edge Laplacian in Chapter 2.1.3, we obtain

\[
\dot{V} \leq 2z^T (I \otimes P) \xi = 2z^T (I \otimes P) (I \otimes A - F \otimes \Gamma) z
\]

\[
= 2x^T (D \otimes I) \left( (I \otimes P) (I \otimes A) - (I \otimes P) (D^T D \otimes \Gamma) \right) (D^T \otimes I) x
\]

\[
= 2x^T \left( (L \otimes PA) - (L \otimes I) (I \otimes PT) (L \otimes I) \right) x
\]

\[
= x^T \left( (L \otimes PA)^T + (L \otimes PA) - (LL \otimes PT)^T - (LL \otimes PT) \right) x.
\]

Since the graph is undirected and connected, there exists an orthogonal matrix \( U \) such that \( U^T LU = \Lambda = \text{diag}(0, \lambda_2, ..., \lambda_N) \), with \( \lambda_i > 0 \) for \( i = 2, ..., N \), and \( U = [U_1, ..., U_N] \) with \( U_1 = 1_N / \sqrt{N} \). Let \( \bar{x} = [\bar{x}_1^T, ..., \bar{x}_N^T]^T = (U^T \otimes I) x \). Then, from (3.11), it follows that

\[
\dot{V} \leq x^T \left( (L \otimes PA)^T + (L \otimes PA) - (LL \otimes PT)^T - (LL \otimes PT) \right) x
\]

\[
= x^T \left( (\Lambda \otimes PA)^T + (\Lambda \otimes PA) - (\Lambda^2 \otimes PT)^T - (\Lambda^2 \otimes PT) \right) \bar{x}
\]

\[
= \sum_{i=2}^{N} \lambda_i \bar{x}_i^T \left( (A - \lambda_i \Gamma)^T P + P (A - \lambda_i \Gamma) \right) \bar{x}_i
\]
If the condition (3.10) is satisfied, we have $\dot{V} \leq 0$ and $\dot{V} = 0$ only when $\bar{x}_i = 0$ for $i = 2, \ldots, N$. Hence, by Lasalle’s invariance principle [47], it follows that $\bar{x}_i \to 0$ as $t \to \infty$ for $i = 2, \ldots, N$. Moreover, since $U_1 = 1_N/\sqrt{N}$, we have $\lim_{t \to \infty} x(t) = \lim_{t \to \infty} (U \otimes I)(U^T \otimes I)x = \lim_{t \to \infty} (\square) x = \frac{1}{N}[\bar{x}_1^T, \ldots, \bar{x}_1^T]^T$, where $\dot{\bar{x}}_1 = A\bar{x}_1$. This completes the proof.

Remark 3.3.1 Recall that for given linear systems, $\dot{x} = A_ix, i = 1, \ldots, N$, $V = x^T P x$ is said to be a common diagonal Lyapunov function (CDLF) if there exists a diagonal positive definite matrix $P$ such that $A_i^T P + PA_i < 0$ for all $i = 1, \ldots, N$. In [48], it was shown that for given linear systems, $\dot{x} = A_ix, i = 1, \ldots, N$, if the Hurwitz matrices $A_i$ are all in upper triangular form, then there always exists a CDLF. Hence, in Theorem 3.3.1, if $A$ is an upper triangular matrix such that $A - \lambda_i\Gamma$ are Hurwitz matrices for all $i = 2, \ldots, N$, then the solution $P$ of Theorem 3.3.1 always exists.

Remark 3.3.2 A matrix $A$ is said to be diagonally stable if there exists a diagonal positive definite matrix $P$ such that $A^T P + PA < 0$. We assume that $\Gamma$ is positive semi-definite, and $A - \lambda_2\Gamma$ is diagonally stable. Then, there exists a diagonal positive definite matrix $P$ such that $(A - \lambda_2\Gamma)^T P + P(A - \lambda_2\Gamma) < 0$. Then, for $i = 3, \ldots, N$, we have

$$
(A - \lambda_i\Gamma)^T P + P(A - \lambda_i\Gamma) < 2(\lambda_2 - \lambda_i)\Gamma P < 0, \quad (3.12)
$$

which satisfies Theorem 3.3.1. The particular classes of diagonally stable matrices are investigated in [49] (see also Remark 2 in [44]).
3.4 Analysis under p-norm bounded interconnection constraints

In this chapter, we consider general constraints, that the constrained sets \( \Omega_k \) for \( k = 1, ..., M \) are defined by the p-norm bounded constraints as follows:

\[
\Omega_k := \left\{ z_k \in \mathbb{R}^n : \|z_k\|_p \leq \Delta, \Delta > 0 \right\},
\]

(3.13)

where p-norm is defined by \( \|z\|_p := \left( \sum_{m=1}^{n} z_{m,k} \right)^{1/p} \). It is clear that the componentwise constraints is a special case with \( p = \infty \). Recall the dynamics of agent \( i \) under interconnection constraints described by (3.4). Since the boundary of the set (3.13) may be coupled with each components of \( z_k \), i.e., \( \partial \Omega_k := \{ z_k \in \mathbb{R}^n : \|z_k\|_p = \Delta \} \), we modify the state saturation \( h(\cdot) \) under the p-norm bounded constraint (3.13) for \( m = 1, ..., n \) as follows:

\[
\dot{z}_{m,k} = h(\xi_{m,k}) :=
\begin{cases} 
0, & \text{if } z_k \in \partial \Omega_k \text{ and } \xi_{m,k} \cdot z_{m,k} \geq 0, \\
\xi_{m,k}, & \text{otherwise.}
\end{cases}
\]

(3.14)

where \( \xi \) is given in (3.4).

**Remark 3.4.1** Note that \( \partial \Omega_k \) could contain the zero components (i.e., \( \exists z_k \in \partial \Omega_k \) satisfying \( z_{m,k} = 0 \) for some \( m \in \{1, ..., n\} \)). Therefore, Lemma 3.3.1 does not hold in general. But, by choosing a diagonal positive definite matrix \( P = \text{diag}(p_1, ..., p_n) \in \mathbb{R}^{n \times n} \), we can conclude that

\[
z^T (I \otimes P) h(\xi) \leq z^T (I \otimes P) \xi.
\]

Then, we can solve the consensus problem under p-norm bounded constraints (3.13) as follows:

**Theorem 3.4.1** Consider an undirected and connected graph \( \mathcal{G} \). The group of \( N \) agents (3.1) under the p-norm bounded constraints (3.13) achieves the consensus for all \( z_k(0) \in \Omega_k \).
\( k = 1, \ldots, M, \) if there exists a diagonal positive definite matrix \( P = \text{diag}(p_1, \ldots, p_n) \in \mathbb{R}^{n \times n} \)

such that

\[
(A - \lambda_i \Gamma)^T P + P (A - \lambda_i \Gamma) < 0, \quad i = 2, \ldots, N,
\]  

(3.15)

where \( \lambda_i \) are the eigenvalues of the Laplacian matrix \( L \).

**Proof:** The proof follows directly from Theorem 3.3.1 and Remark 3.4.1 with the diagonal positive definite matrix \( P = \text{diag}(p_1, \ldots, p_n) \).

\[ \blacksquare \]

### 3.5 Applications

#### 3.5.1 Rendezvous problem under limited communication ranges and input constraints

In this chapter, as an example of application of our framework, we study the rendezvous problem under limited communication ranges and input constraints. We consider \( N \) single integrator agents in two-dimensional space

\[
\dot{x}_i = u_i, \quad i \in \mathcal{V} := \{1, \ldots, N\},
\]  

(3.16)

where \( x_i = [x_{1,i}, x_{2,i}]^T \in \mathbb{R}^2 \) and \( u_i = [u_{1,i}, u_{2,i}]^T \in \mathbb{R}^2 \) are the position and the control input vector of agent \( i, \) \( i \in \mathcal{V}, \) respectively. We assume that there are \( M \) edges, and the \( k \)-th edge state is defined as \( z_k = [z_{1,k}, z_{2,k}]^T \in \mathbb{R}^2, \) \( k = 1, \ldots, M \).

Distance between two arbitrary agents \( i \) and \( j \) connected by an edge \( k \) is defined by 2-norm, i.e.,

\[
\|z_k\| = \sqrt{(x_i - x_j)^T (x_i - x_j)} \geq 0.
\]
In practice, the communication constraint between two agents \(i\) and \(j\) is given by the distance between them. Thus, the constrained sets \(\Omega_k\) for \(k = 1, \ldots, M\) are defined by

\[
\Omega_k := \{ z_k \in \mathbb{R}^2 : \|z_k\| \leq \Delta, \Delta > 0 \}. \tag{3.17}
\]

We assume that each agent measures own position and velocity, and transmits the measurements to their neighbors if and only if the agents are within their communication range, i.e., \(z_k \in \Omega_k\). We further assume that the input of each agent is subject to input constraints, i.e., \(|u_i| \leq \bar{u}\). Then, based on the model of the \(i\)th system (3.4) under interconnection constraint (3.14) with \(A = 0\) and \(\Gamma = I\), we propose the following consensus algorithm:

\[
\begin{align*}
\dot{u} &= -(KD \otimes I)z, \\
\dot{z} &= h(\xi), \\
\xi &= -(D^T KD \otimes I)z, \tag{3.18}
\end{align*}
\]

where \(u = [u_1^T, \ldots, u_N^T]^T\), \(z = [z_1^T, \ldots, z_M^T]^T\), \(\xi = [\xi_1^T, \ldots, \xi_M^T]^T\), \(K = \text{diag}(k_1, \ldots, k_N)\) with \(k_i > 0\) for all \(i = 1, \ldots, N\) is a gain matrix which will be determined later, and \(h(\cdot)\) is defined in (3.14). Then, we first show that the connectedness is preserved under the proposed consensus algorithm as follows:

**Lemma 3.5.1** Consider the consensus algorithm (3.18) with the communication constraints (3.17). If, for all \(k = 1, \ldots, M\), \(z_k(0) \in \Omega_k\), then \(z_k(t) \in \Omega_k\) for all \(t \geq 0\).

**Proof:** Consider the time derivative of \(\|z_k\|^2\) as

\[
\frac{d}{dt} \|z_k\|^2 = 2z_k^T \dot{z}_k = 2 (z_{1,k} \dot{z}_{1,k} + z_{2,k} \dot{z}_{2,k}),
\]

where \(z_{1,k} = \dot{z}_k - D z_k\) and \(z_{2,k} = \dot{\xi}_k = (K \otimes I) z_k\).
which has the same sign as \( \frac{d}{dt} \|z_k\| \) for all \( \|z_k\| > 0 \) but defined on all of \( \mathbb{R}^2 \). Hence, a sufficient condition for two connected agents to remain connected for all \( t \geq 0 \) is \( \frac{d}{dt} \|z_k\|^2 \leq 0 \) when \( \|z_k\| = \Delta \). Then, under the consensus algorithm (3.18) with \( z_k \in \partial \Omega_k \), \( \frac{d}{dt} \|z_k\|^2 = 0 \) if \( z_{1,k} \cdot \xi_{1,k} > 0 \) and \( z_{2,k} \cdot \xi_{2,k} > 0 \), and \( \frac{d}{dt} \|z_k\|^2 < 0 \), otherwise. Therefore, the set \( \Omega_k \) for \( k = 1, \ldots, M \) is invariant.

Lemma 3.5.1 illustrates the invariance of the constrained set \( \Omega_k \) for \( k = 1, \ldots, M \) under proposed control. Then, we are now in a position to state the rendezvous result:

**Corollary 3.5.1** Consider an undirected and connected graph \( G \). For all \( z_k(0) \in \Omega_k, k = 1, \ldots, M \), the group of \( N \) agents (3.16) with the consensus algorithm (3.18) asymptotically converges to the weighted centroid \( \bar{x} = \frac{\sum_{i=1}^{N} k_i^{-1} x_i(0)}{\sum_{i=1}^{N} k_i^{-1}} \) while preserving the connectedness.

**Proof:** Let \( x = [x_1^T, \ldots, x_N^T]^T, z = [z_1^T, \ldots, z_M^T]^T, \xi = [\xi_1^T, \ldots, \xi_M^T]^T, \) and \( V = \frac{1}{2} z^T z \).

Then, following the proof of Theorem 3.3.1 and Remark 3.4.1 we can easily obtain

\[
\dot{V} \leq z^T \xi = -z^T (D^T KD \otimes I) z
\]

\[
= -x^T (D \otimes I)(D^T K D \otimes I) D^T \otimes I) x
\]

\[
\leq - \sum_{i=1}^{2} \hat{x}_i^T LKL \hat{x}_i,
\]

where \( \hat{x}_i = [x_{i,1}, \ldots, x_{i,N}]^T, i = 1, 2 \). Since \( K \) and \( L \) are the positive and the positive semi-definite, respectively, \( LKL \) is positive semi-definite. Hence, \( \dot{V} \leq 0 \) for all \( z_k \in \Omega_k, k = 1, \ldots, M \). Furthermore, according to Lemma 3.5.1 the constrained set \( \Omega_k \) for all \( k = 1, \ldots, M \), is invariant, and \( \text{span}\{1\} \) is \( L \)-invariant. Hence, \( \dot{V} = 0 \) only when \( \hat{x}_i \in \text{span}\{1\} \), and, then, by LaSalle’s invariance principle, it follows that \( \hat{x}_i \) converges to \( \text{span}\{1\} \). Next, we
will show that the agents converge to the weighted centroid given by \( \bar{x} = \frac{\sum_{i=1}^{N} \sum_{j=1}^{k_i-1} x_i}{\sum_{j=1}^{k_i-1}} \). The componentwise dynamics of \( \bar{x} \) is given by

\[
\frac{d}{dt} \bar{x}_i = \frac{1^T K^{-1}}{\sum_{j=1}^{N} k_j} \frac{d}{dt} \bar{x}_i = -\frac{1^T K^{-1} K D}{\sum_{j=1}^{N} k_j} \dot{\bar{x}}_i = 0,
\]

where \( \dot{\bar{x}}_i = [x_{i,1}, \ldots, x_{i,M}]^T \). Therefore, the weighted centroid is invariant. Let the weighted centroid be given by \( \bar{x}_i = \frac{1^T K^{-1} \bar{x}_i(0)}{\sum_{j=1}^{N} k_j} = C \). Furthermore, since the \( \dot{\bar{x}}_i \) converges to \( \text{span}\{1\} \), there exists \( \bar{C} \) such that \( \lim_{t \to \infty} \dot{\bar{x}}_i(t) = \bar{C} \mathbf{1} \) for some \( \bar{C} \). Then, since the weighted centroid is invariant, we have

\[
\bar{x}_i = C = \frac{1^T K^{-1} \bar{C} \mathbf{1}}{\sum_{j=1}^{N} k_j} = \bar{C} \frac{1^T K^{-1} \mathbf{1}}{\sum_{j=1}^{N} k_j} = \bar{C}.
\]

Therefore, we have \( \bar{C} = C \), which implies the agents converge to the weighted centroid.

We next construct the control gain \( K \) such that the input constraints are guaranteed. Since the constrained set \( \Omega_k \) is invariant, the edge states also satisfy the componentwise constraints, i.e., \( |z_k| \leq \Delta \), and thus the control input is also bounded as \( |u_i| \leq k_i d_i \Delta \), where \( d_i = \sum_{k=1}^{M} |d_{ik}| \) denotes the number of neighbors of agent \( i \). Therefore, by choosing the control gain \( k_i \) such that \( |u_i| \leq k_i d_i \Delta \leq \bar{u} \), the proposed algorithm guarantees consensus with any input constraints.

For the numerical example, a group of \( N = 9 \) agents with communication constraint \( \Delta = 4.5 \) is considered. The initial positions and the network topology are given in Fig. 3.1. It was shown in [30] that even though the initial graph is connected, the connectedness is lost when the standard consensus algorithm \( u_i = \sum_{j=1, j \neq i}^{N} \alpha_{ij} (x_j - x_i) \) is used.

We first consider the input constraints as \( |u_i| \leq \bar{u} = 4.5 \) for all \( i = 1, \ldots, 9 \). Then, considering the network topology and the input constraints, we choose the control gain \( K = \)
Figure 3.1: Initial network topology.

blkdiag(0.3 * I_{3\times3}, 0.25, 0.5, 0.25, 0.3 * I_{3\times3}). In Fig. 3.2, the evolution of the agent group under the proposed consensus algorithm (3.18) are depicted. Since the weighted centroid equals to the origin, as we can see from this figure, the agents converge to the origin while preserving the connectedness. Furthermore, Fig. 3.3 shows that each agent also satisfies the input constraints.

We next consider the input constraints as \(|u_i| \leq \bar{u}_i = 2\) for \(i = 1, ..., 4\) and \(|u_i| \leq \bar{u}_i = 4.5\) for \(i = 5, ..., 9\). We choose the control gain \(K = \text{blkdiag}(0.1* I_{4\times4}, 0.5, 0.25, 0.3* I_{3\times3})\). In this case, the weighted centroid is given by \(\bar{x} = [-2.8, 0]^T\). Fig. 3.4 shows the trajectories of each agents, and the convergence to the weighted centroid. Furthermore, as we can see from the Fig. 3.5, the agents satisfy the input constraints \(|u_i| \leq 2\) for \(i = 1, ..., 4\) (dashed line) and \(|u_i| \leq 4.5\) for \(i = 5, ..., 9\).
Figure 3.2: Evolution of the agents \((\Delta = 4.5, \bar{u} = 4.5)\).

Figure 3.3: Inputs of the agents \((\Delta = 4.5, \bar{u} = 4.5)\).
Figure 3.4: Evolution of the agents ($\Delta = 4.5$, $\bar{u}_i = 2$ for $i = 1, \ldots, 4$, $\bar{u}_i = 4.5$ for $i = 5, \ldots, 9$).

**Remark 3.5.1** If the network topology is determined from the connectivity among agents, which is defined by communication range, then the topology will be maintained under the proposed algorithm.

### 3.5.2 Load balancing problem under flow constraints

In this chapter, we study the load balancing of storage devices under flow constraints. Consider a group of $N$ storage devices as follows:

$$\dot{x} = Du,$$  \hspace{1cm} (3.19)

where $x \in \mathbb{R}^N$ and $u \in \mathbb{R}^M$ are the total amount of resource stored in the storage devices and the resource flows, respectively. Then, the goal of the load balancing is to achieve an
Figure 3.5: Inputs of the agents ($\Delta = 4.5$, $\bar{u}_i = 2$ for $i = 1, ..., 4$ (dashed line), $\bar{u}_i = 4.5$ for $i = 5, ..., 9$ (solid line)).

Figure 3.6: Liquid level control systems.
agreement to a common state, i.e., \( \lim_{t \to \infty} x_i - x_j = 0, \forall i, j \in V \).

**Remark 3.5.2** Note that in [50], it was shown that the load balancing solves the capacity maximization problem.

In real applications, the flow between the storage devices are constrained due to the physical limitations. For example, Liquid level systems consist of several liquid tanks and their connections, i.e., see, Fig. 3.6 [51]. The liquid tanks are connected through the pipes which imply the flows between liquid tanks, and thus, the flows are constrained due to the maximum capacities of pipes. Therefore, the goal is to achieve an agreement of the storage devices under the flow constraints.

We consider the following PI control with flow constraints [46]:

\[
\dot{\zeta} = D^T x \\
\dot{u} = \text{sat}(-D^T x - \zeta),
\]

(3.20)

where \( \text{sat}(\cdot) = \text{sign}(\cdot) \max\{|\cdot|, \Delta\}, \Delta > 0 \), is the saturation function, which represents the flow constraints. Then, the flow dynamics can be written in terms of the state saturation as follows:

\[
\dot{u} = h(-D^T \dot{x} - \dot{\zeta}) \\
= h(-D^T Du - D^T x).
\]

(3.21)

We choose the following Lyapunov function candidate:

\[
V = \frac{1}{2}(u^T u + x^T x).
\]

(3.22)
Then, the derivative of $V$ is given by

$$\dot{V} = u^T h(-D^T Du - D^T x) + x^T Du,$$

(3.23)

and following the proof of Theorem 3.3.1, we have

$$\dot{V} \leq -u^T D^T Du - u^T D^T x + x^T Du$$

$$= -u^T D^T Du,$$

(3.24)

which implies $\dot{V} \leq 0$. Let $\mathcal{M} := \{u \in \mathbb{R}^m : Du = 0\}$. Then, by LaSalle Invariance principle, any solution of $u_i$ will converge to the largest invariant set inside $\mathcal{M}$. In the invariant set within $\mathcal{M}$, $\dot{x} = 0$, which implies $x(t) = x^*$ for some constant vector $x^*$. Moreover, $\dot{u} = h(-D^T Du - D^T x^*) = h(-D^T x^*) = h(z^*)$. Assume that $z^* \neq 0$. Then, $u_i = -1$ if $z_i^* > 0$, $u_i = 1$ if $z_i^* < 0$, and $|u_i| < 1$ otherwise. Therefore, $\dot{u} = 0$, and it follows that

$$u^T u = u^T h(-z^*) \leq -u^T z^* = 0,$$

(3.25)

which gives $-u^T z^* = 0$. However, if $z^* \neq 0$, then $-u^T z^* > 0$, which is a contradiction. Therefore, By LaSalle Invariance principle, we have $\lim_{t \to \infty} z(t) = \lim_{t \to \infty} D^T x = 0$. Since the graph is connected, the null space of $D^T$ is spanned by $\{1\}$. Therefore, we can conclude that $\lim_{t \to \infty} x_i = C$, $\forall i \in \mathcal{V}$, $C \in \mathbb{R}$. Moreover, the average value is invariant, it is clear that $C = \frac{1}{N} \sum_{i=1}^{N} x_i(t_0)$.

We next consider a group of $N$ storage devices with supply and demand as follows:

$$\dot{x} = Du + Ed,$$

(3.26)
where \( d \in \mathbb{R}^K \) implies inflows and outflows, and \( E \in \mathbb{R}^{N \times K} \) specifies the vertices where flow can enter or leave the network. Then, under the flow control (3.20) with flow constraints, the flow dynamics is given by

\[
\dot{u} = h(-D^T \dot{x} - \zeta) = h(-D^T Du - D^T Ed - B^T x), \tag{3.27}
\]

Assume that there exists a matching controller state, i.e.,

\[
Ed = D\zeta^*, \tag{3.28}
\]

with \(|\zeta^*| \leq \Delta\). Then, before we prove the load balancing, we extend Lemma 3.3.1 as follows:

**Lemma 3.5.2** Let \( \xi = -D^T Du - D^T Ed - B^T x \). Then, we have

\[
(u + \zeta^*)h(\xi) \leq u^T \xi. \tag{3.29}
\]

**Proof:** Following the proof of Lemma 3.3.1, we have

\[
(u + \zeta^*)(\xi - h(\xi)) = \sum_{i=1}^{M} (u_i + \zeta^*_i)(\xi - h(\xi_i)) = \sum_{i \in S} (u_i + \zeta^*_i)\xi_i. \tag{3.30}
\]

Note that \( S \) represents the set of saturated components, and thus, \(|u_i| = \Delta \) and \( u_i \cdot \xi_i > 0 \), \( \forall i \in S \). Since \(|\zeta^*| \leq \Delta\), \( (u_i + \zeta^*_i)\xi_i \geq 0 \), \( \forall i \in S \), which complete the proof.

Let \( \bar{u} = u + \zeta^* \), and define the following Lyapunov function candidate:

\[
V = \frac{1}{2}(\bar{u}^T \bar{u} + x^T x). \tag{3.31}
\]
Then, the derivative of $V$ with Lemma 3.5.2 is given by

\[
\dot{V} = \bar{u}^T h(-D^T \bar{D} u - D^T x) + x^T (D u + E d)
\]

\[
\leq - \bar{u}^T D^T \bar{D} u - \bar{u}^T D^T x + x^T D u + x^T E d
\]

\[
= - \bar{u}^T D^T \bar{D} u - \bar{u}^T D^T x + x^T D \bar{u}
\]

\[
\leq 0.
\]

Then, following the same procedure as the case without supply and demand, we can conclude that

\[
\lim_{t \to \infty} x_i(t) = C, \forall i \in \mathcal{V}.
\]

For the numerical example, a group of $N = 6$ agents with flow constraints $\Delta = 1$ is considered. The graph topology and its incidence matrix are given in Fig. 3.7.

We first consider the agents without supply and demand. Then, the simulation results are given in Fig. 3.8 and Fig. 3.9. Although the flows are constrained as we can see from Fig. 3.9, the agents converge to the average value, i.e., \( \lim_{t \to \infty} x_i(t) = \frac{1}{N} \sum_{i=1}^{N} x_i(t_0) = 3 \).

We next consider the supply $d_1 = 0.5$ and the demand $d_2 = -0.5$. The simulation results are given in Fig. 3.10 and Fig. 3.11. As we can see from Fig. 3.10, the agents converge to the common state. Moreover, since the supply and the demand are balanced, i.e., $d_1 + d_2 = 0$, we can easily see that the average value is invariant, i.e., $\frac{1}{N} \sum_{i=1}^{N} \dot{x}_i = 0$. Therefore, the agents converge to the average value, and we know from Fig. 3.11 that the flows converge to the matching state, i.e., $D \zeta^* = D u = E d$. 

\[\text{– 36 –}\]
In this chapter, the analysis and synthesis problem for consensus of a group of systems under interconnection constraints are studied. Since the constraints in the state variables can be modeled with state saturations, the systems under interconnection constraints are governed by the edge dynamics with state saturations. Then, sufficient conditions are identified.
under which the given constrained systems achieve the consensus. By utilizing the state saturation, we construct the nonlinear consensus algorithm that solves the rendezvous problem while preserving the connectedness and the input constraints. Moreover, the analysis technique is applied to analyze the balancing problem under flow constraints.
Figure 3.10: Evolution of the agents with supply $d_1 = 0.5$ and demand $d_2 = -0.5$.

Figure 3.11: Evolution of the flows with supply $d_1 = 0.5$ and demand $d_2 = -0.5$. 

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Chapter 4

Consensus with output saturations

4.1 Introduction

In this chapter, we consider a consensus problem under output saturations. In the consensus problem setup, each agent measures its own state, and exchanges this information with its neighbors such that the states of all agents converge to a certain value. Consider a group of $N$ single-integrator modeled agents, and let $x_i, y_i \in \mathbb{R}$ be the state and the measured output of agent $i$. Then, a standard consensus algorithm takes the following form \[4\]:

$$\dot{x}_i = \sum_{j=1}^{N} \alpha_{ij}(t)(y_j - y_i), \quad i \in \mathcal{V} := \{1, \ldots, N\}, \quad (4.1)$$

and then, the overall networked agent has the form

$$\dot{x} = -L(t)y. \quad (4.2)$$

For an undirected graph, the null space of the Laplacian matrix $L(t)$ is $\text{span}\{1\}$. Therefore, an equilibrium of (4.2) is a state in the form $y^* = C1$, $C \in \mathbb{R}$. Assume that each agent can measure the exact state, i.e., $y_i = x_i \forall i \in \mathcal{V}$. Then, when the fixed graph is connected \[2–4\] or the time-varying graph is integrally connected over $[0, \infty)$ \[39\], $y^* = x^* = C1$ is an unique equilibrium that implies the agents (4.1) achieve the consensus.

Meanwhile, in real applications, the usage of the measurement units may lead to some nonlinearities over the network. For example, due to the digital communication channels
or digital sensors, the consensus problems under quantization effects have been studied for a fixed graph in [52] and a time-varying graph in [53]. The consensus has been derived by utilizing some properties of the Laplacian [52] and the integral graph [53]. In [54–56], the consensus problems have been studied for more general nonlinearities with (strictly) increasing or decreasing conditions. In the above results, the nonlinearities were assumed to be unbounded.

On the other hand, the output saturations, that is a bounded nonlinearity, arises frequently due to the sensor limitations. While the consensus problem under input saturations has been addressed in much detail [13,34–36], the output saturations have been received fewer results [37, 38]. Note that, in [38, 55], it was pointed out that for the bounded nonlinearity, the consensus may not be realized due to the existence of stable, unachievable equilibrium points for the consensus (see, Remark 3 in [55]). Let us consider a simple example when \( y_i = \text{sat}(x_i) \) with \( \text{sat}(\cdot) = \text{sign}(\cdot) \max\{|\cdot|, 1\} \). In this case, an equilibrium of (4.2) is a state in the form \( \text{sat}(x^*) = C1 \). Thus, the set of equilibria of (4.2) can be divided into two groups as

\[
\Omega_a := \{x \in \mathbb{R}^N : x = C1, |x| \leq 1 \} \quad \text{and} \quad \Omega_u := \{x \in \mathbb{R}^N : x > 1 \text{ or } x < -1 \}.
\]

It is clear that \( \Omega_a \) is the set of achievable equilibrium, i.e., \( x^* \in \Omega_a \) implies that the consensus is reached, but \( \Omega_u \) may not (see, Chapter 4.4.3). Therefore, under the bounded constraints, [37,38] have developed the consensus algorithms. Specifically, in [37], the discarded consensus algorithm, which discards the state of a neighbor if the state is outside its constraints, was proposed. In [38], the output feedback based leader-following consensus algorithm was studied.

Under the standard consensus algorithm, the agents converge to the average value. However, as mentioned above, the consensus under the standard setup may not be achieved under
output saturations. Although some results have been available for the consensus problem under output saturations, an analytic result has not been achieved. Therefore, this chapter investigates conditions for achieving consensus under output saturations.

This chapter considers the dynamics of each agent as a single-integrator, and both fixed and time-varying undirected graphs. Moreover, we consider homogeneous and heterogeneous saturations levels, in which the agents have identical and different saturation levels, respectively. Then, we first analyze the consensus under the fixed and connected graph. By utilizing an integral Lyapunov function, we investigate necessary and sufficient conditions for achieving the consensus. We next consider the consensus under the time-varying graph topology with an integrally connected condition, which is the necessary and sufficient graph condition for achieving the consensus. We first analyze an attractivity of equilibrium. Then, we derive the necessary and sufficient conditions by investigating conditions for the achievable equilibrium. Moreover, we extend the results to the cases of double-integrator modeled agents, and the directed graph.

Sequentially, the main contributions of this chapter are as follows: (1) Under the standard consensus algorithm, we prove an asymptotic convergence of agents with output saturations. We consider general saturation levels and graph topology. The analysis techniques of this chapter rely on the strictly increasing property of the saturation function within its bounds. Thus, the analysis can be easily extended to any bounded nonlinearities, which are strictly increasing within its bounds. (2) We investigate some properties of the set of equilibria. Then, necessary and sufficient conditions for achieving the consensus are obtained. (3) The analytic results are extended to the cases of double-integrator modeled agents as well as the
fixed and directed graph cases.

4.2 Problem statement

In this chapter, we consider a group of $N$ single-integrator modeled agents under output saturations, and the following standard consensus algorithm:

$$\dot{x}_i = \sum_{j=1}^{N} \alpha_{ij}(t)(y_j - y_i), \quad i \in \mathcal{V} := \{1, \ldots, N\},$$

$$y_i = \text{sat}_i(x_i), \quad (4.3)$$

where $x_i, y_i \in \mathbb{R}$ are the state and measured output of the agent $i$, and the saturation function is defined as

$$\text{sat}_i(x_i) = \text{sign}(x_i) \max\{|x_i|, s_i\}, \quad s_i > 0, \quad (4.4)$$

where $s_i$ presents the saturation level, and we use $\text{sat}_i(x_i) = \text{sat}(x_i)$ for $s_i = s$, $\forall i \in \mathcal{V}$. Then, we say that the agents are homogeneous if $y_i = \text{sat}(x_i), \forall i \in \mathcal{V}$, and heterogeneous otherwise.

This chapter studies the consensus problem for the $N$ agents with output saturations. The consensus is said to be achieved for the group of $N$ agents if $\lim_{t \to \infty} x_i = C, \forall i \in \mathcal{V}$, where $x^*$ is called the group decision value. As mentioned in the introduction section, the overall networked agent (4.2) contains unachievable equilibrium points under output saturations. Thus, this chapter will prove that, under certain initial conditions, the agents (4.3) achieve the consensus.

In this chapter, we consider the following assumption to avoid the trivial solution.
Assumption 4.2.1 Without loss of generality, there exists \( i, j \in V, i \neq j \), such that \( x_i(t_0) \neq x_j(t_0) \).

4.3 Undirected and Fixed Graph

This chapter deals with the consensus problem under the undirected and fixed graph. We first investigate some properties of agents (4.3) under the undirected and fixed graph.

Lemma 4.3.1 Consider the group of \( N \) agents (4.3), and suppose the graph is undirected and fixed. Then, the average of all agent states \( \frac{1}{N} \sum_{i=1}^{N} x_i(t) \) is invariant, \( \forall t \geq t_0 \).

Proof: The time derivative of the average value is given by \( \frac{1}{N} \sum_{i=1}^{N} \dot{x}_i(t) = \frac{1}{N} 1^T \dot{x} = -\frac{1}{N} 1^T Ly = 0 \). Therefore, the average value is preserved, \( \forall t \geq t_0 \).

Remark 4.3.1 From Lemma 4.3.1, it is clear that the group decision value \( C \) satisfies
\[
\lim_{t \to \infty} \frac{1}{N} \sum_{i=1}^{N} x_i(t) = C.
\]

Lemma 4.3.2 For an undirected graph and any \( x_i, y_i \in \mathbb{R}, i = 1, ..., N \), we have
\[
\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_{ij} (x_i - x_j)(y_j - y_i) = 2 \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_{ij} x_i (y_i - y_j). \tag{4.5}
\]

Then, we are now ready to state the following result.

Theorem 4.3.1 Suppose the graph is undirected and connected. Then, the group of \( N \) agents (4.3) achieves the consensus, if and only if \( \frac{1}{N} \left| \sum_{i=1}^{N} x_i(t_0) \right| \leq \min_{i \in V} \{ s_i \} \).

Proof: (Necessity) We prove the necessity by a contradiction. Assume that the agents achieve the consensus with \( \frac{1}{N} \sum_{i=1}^{N} x_i(t_0) > \min_{i \in V} \{ s_i \} \), which gives the group decision
value is given by $C > \min_{i \in V} \{s_i\}$. Then, from Assumption 4.2.1, there exists a time $t^* \geq t_0$ such that $x_i(t^*) \geq \min_{i \in V} \{s_i\}, \forall i \in V$, and the consensus is not reached. Then, the proof is divided into two cases.

1) Homogeneous agents.

Note that $x_i(t^*) \geq s$ implies $y_i(t^*) = s$. Therefore, the agent $i, \forall i \in V$, is given by for $t \geq t^*$

$$\dot{x}_i = \sum_{j=1}^{N} \alpha_{ij} (y_j - y_i) = 0, \quad (4.6)$$

which gives $\lim_{t \to \infty} x_i(t) = x_i(t^*)$. However, we have assumed that $x_i(t)$ at $t^*$ is not reached the consensus, this is a contradiction.

2) Heterogeneous agents.

Let $i' = \text{arg}\min_{i \in V} \{s_i\}$. Then, $x_i(t^*) \geq s_i', \forall i \in V$, implies $y_{i'}(t^*) \geq s_i'$ and $y_j(t^*) \geq s_i', \forall j \in V \setminus \{i'\}$. Therefore, the agent $i'$ is given by for $t \geq t^*$

$$\dot{x}_{i'} = \sum_{j=1}^{N} \alpha_{ij} (y_j - y_{i'}) \geq 0, \quad (4.7)$$

and $x_{i'} = 0$ only when $y_j = s_{i'}, \forall j \in N_{j'}$. Since the graph is connected, the consensus is reached only when $x_i = s_{i'}, \forall i \in V$, that is $\lim_{t \to \infty} x_i(t) = x^* = s_{i'} = 1/N \sum_{i=1}^{N} x_i(t)$. Since the average value is invariant, this is a contradiction.

For $\frac{1}{N} \sum_{i=1}^{N} x_i(t_0) < \min_{i \in V} \{s_i\}$, we can similarly derive that when $x_i(t^*) \leq \max_{i \in V} \{-s_i\}, \forall i \in V$, and the consensus is not reached, 1) $\dot{x}_i = 0$ for the homogeneous agents, and 2) $\dot{x}_{i'} \leq 0$ for the heterogeneous agents. Then, the following the same process as above, we can derive the necessity, which completes the proof.
(Sufficiency) Let \( x^* = \frac{1}{N} \sum_{i=1}^{N} x_i(t_0) \), and \( i' = \arg\min_{i \in V} \{ s_i \} \), and assume that \( |x^*| \leq s_i \). Consider the following Lyapunov function candidate:

\[
V = 2 \sum_{i=1}^{N} \int_{x^*}^{x_i} \text{sat}_i(\tau) d\tau. \tag{4.8}
\]

We first show that \( V \geq 0 \) by subtracting \( 2 \sum_{i=1}^{N} \int_{x^*}^{x_i} x^* d\tau \) from the both side of (4.8). Note that, since \( x^* = \frac{1}{N} \sum_{i=1}^{N} x_i \)

\[
\sum_{i=1}^{N} \int_{x^*}^{x_i} x^* d\tau = x^* \sum_{i=1}^{N} x_i - \sum_{i=1}^{N} x^* x_i^2 = 0. \tag{4.9}
\]

Then, by subtracting \( 2 \sum_{i=1}^{N} \int_{x^*}^{x_i} x^* d\tau \) from the both side of (4.8), we have

\[
V = 2 \sum_{i=1}^{N} \int_{x^*}^{x_i} (\text{sat}_i(\tau) - x^*) d\tau. \tag{4.10}
\]

Then, the right-hand side of (4.10) gives

\[
2 \sum_{i=1}^{N} \int_{x^*}^{x_i} (\text{sat}_i(\tau) - x^*) d\tau = 2 \sum_{i=1}^{N} \left( \frac{1}{2} \left( x_i^2 - x^* x_i^2 - (x_i - \text{sat}_i(x_i))^2 \right) - x^* x_i + x^* x_i^2 \right)
\]

\[
= \sum_{i=1}^{N} \left( x^* x_i^2 - \text{sat}_i(x_i)^2 + 2x_i \text{sat}_i(x_i) - 2x^* x_i \right)
\]

\[
\geq \sum_{i=1}^{N} \left( \text{sat}_i(x_i)^2 + x^* x_i^2 - 2x^* x_i \right)
\]

\[
= \sum_{i=1}^{N} \left( \text{sat}_i(x_i) - x^* \right)^2, \tag{4.11}
\]

which implies \( V \geq 0 \). We next consider the time derivative of \( V \) (4.8). Since the average value is invariant, i.e., \( \dot{x}^* = 0 \), \( \dot{V} \) is given by

\[
\dot{V} = 2 \sum_{i=1}^{N} \text{sat}_i(x_i) \dot{x}_i
\]

\[
= 2 \sum_{i=1}^{N} y_i \sum_{j=1}^{N} \alpha_{ij} (y_j - y_i). \tag{4.12}
\]
Then, from Lemma 4.3.2 it follows that

\[ \dot{V} = -\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_{ij} (y_i - y_j)^2, \]  

(4.13)

which implies \( \dot{V} \leq 0 \). Let \( \mathcal{M} := \{ x \in \mathbb{R}^N : \dot{V} = 0 \} \). Then, by LaSalle Invariance Principle, any solution of \( x_i(t), \forall i \in \mathcal{V} \), will converge to the largest invariant set inside \( \mathcal{M} \). We next prove that \( x_i = x_j, \forall i, j \in \mathcal{V} \), is an unique equilibrium, which implies the consensus is reached. Since the graph is connected, \( \dot{V} \equiv 0 \) implies that \( (y_i - y_j) = (\text{sat}_i(x_i) - \text{sat}_j(x_j)) = 0, \forall i, j \in \mathcal{V} \). Then, the rest of the proof is divided into two cases.

1) Homogeneous case.

Note that \( (\text{sat}(x_i) - \text{sat}(x_j)) \equiv 0, \forall i, j \in \mathcal{V} \), when (a) \( x_i \geq s \) or \( x_i \leq -s, \forall i \in \mathcal{V} \), (b) \( x_1 = \cdots = x_N \) with \( |x_i| \leq s, \forall i \in \mathcal{V} \). Since the average value is invariant, if \( \frac{1}{N} \left| \sum_{i=1}^{N} x_i(t_0) \right| \leq s \), then the case (a) cannot be realized, \( \forall t \geq t_0 \). Therefore, \( (\text{sat}(x_i) - \text{sat}(x_j)) \equiv 0, \forall i, j \in \mathcal{V} \), only when \( (x_i - x_j) \equiv 0, \forall i, j \in \mathcal{V} \).

2) Heterogeneous case.

Since \( |\text{sat}_{i'}(x_{i'}(t))| \leq s_{i'}, \forall t \geq t_0 \), where \( i' = \text{argmin}_{i \in \mathcal{V}} \{ s_i \} \), and the condition \( (\text{sat}_i(x_i) - \text{sat}_j(x_j)) \equiv 0, \forall i, j \in \mathcal{V} \), is equivalent to \( (\text{sat}_{i'}(x_{i'}) - \text{sat}_j(x_j)) \equiv 0, \forall j \in \mathcal{V} \). This condition is satisfied when (a) \( x_{i'} > s_{i'} \) and \( x_j = s_{i'} \) or \( x_{i'} < -s_{i'} \) and \( x_j = -s_{i'} \), \( \forall j \in \mathcal{V} \setminus \{ i' \} \), (b) \( x_1 = \cdots = x_N \) with \( |x_i| \leq s_{i'}, \forall i \in \mathcal{V} \). If \( \frac{1}{N} \left| \sum_{i=1}^{N} x_i(t_0) \right| \leq s_{i'} \), the cases (a) cannot be realized, and thus, \( (\text{sat}_{i'}(x_{i'}) - \text{sat}_j(x_j)) \equiv 0, \forall j \in \mathcal{V} \setminus \{ i' \} \), only when \( (x_i - x_j) \equiv 0, \forall i, j \in \mathcal{V} \).

In summary, we have shown that \( \dot{V} \leq 0 \) and \( \dot{V} \equiv 0 \) only when \( (x_i - x_j) \equiv 0, \forall i, j \in \mathcal{V} \). Therefore, according to LaSalle Invariance Principle, we have \( \lim_{t \to \infty} (x_i(t) - x_j(t)) = 0, \forall i, j \in \mathcal{V} \), which complete the proof. \( \blacksquare \)
4.4 Undirected and Time-varying Graph

In this chapter, we consider a time-varying graph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), \mathcal{A}(t))$ with the following assumption:

**Assumption 4.4.1** For $\forall i, j \in \mathcal{V}$, $\alpha_{ij}(t)$ is a continuous function on $[0, \infty)$ except for at most a set with measure zero.

Then, under Assumption 4.4.1, the set of discontinuity points for the right-hand side of (4.3) has measure zero. Therefore, the Caratheodory solutions\(^1\) of (4.3) exist for arbitrary initial conditions, which satisfies for all $t \geq t_0$ the following integral equation for $i \in \mathcal{V}$:

$$x_i(t) = x_i(t_0) + \int_{t_0}^{t} \sum_{j=1}^{N} \alpha_{ij}(\tau)(y_j(\tau) - y_i(\tau))d\tau.$$  \hspace{1cm} (4.14)

Before we analyze the consensus under the time-varying graph, we introduce some mathematical preliminaries. Since the time-varying graph include discontinuities to describe the switching phenomena, the solution of $x_i(t)$ is not differentiable at the discontinuous points. However, from Assumption 4.4.1, the upper Dini derivative of $x_i$ along the solution exists. The upper Dini derivative of a function $f : (a,b) \rightarrow \mathbb{R}$ at $t$ is defined as

$$D^+ f(t) = \limsup_{\tau \to 0^+} \frac{f(t + \tau) - f(t)}{\tau}.$$  \hspace{1cm} (4.15)

**Lemma 4.4.1** \(^{[59]}\) Suppose $f(t)$ is continuous on $(a,b)$. Then, $f(t)$ is nonincreasing on $(a,b)$ if and only if $D^+ f(t) \leq 0$, $\forall t \in (a,b)$.

\(^1\)Caratheodory solutions are a generalization of classical solutions, and absolutely continuous functions of time. Caratheodory solutions relax the classical requirement that the solution must follow the direction of the vector field at all times, see \(^{[57],[58]}\) for details.
4.4.1 Homogeneous agents

We first analyze the consensus of the agents (4.3) under the homogeneous condition, i.e., $s_i = s, \forall i \in \mathcal{V}$. To solve this problem, we use the notations used in [39] as follows.

For any time $t$, let $M_k(t)$ be the $k$-th largest value of the components $x_i(t)$, that is, we rank $x_i(t)$ with descending order for each $t$ as follows:

$$x_{i_1}(t) \geq x_{i_2}(t) \geq \cdots \geq x_{i_N}(t),$$  \hspace{1cm} (4.16)

where $\{i_1, \ldots, i_N\}$ is a permutation of $\{1, \ldots, N\}$, and define

$$M_k(t) = x_{i_k}(t).$$  \hspace{1cm} (4.17)

Note that the permutation $\{i_k : k \in \mathcal{V}\}$ depends on $t$, and the permutation $i_k$ is piecewise constant. As a result, $M_k(t)$ is absolutely continuous for almost everywhere. We further denote $S_k(t)$ as the sum of the first $k$ largest values of $x_i(t)$, i.e.,

$$S_0(t) = 0, \quad S_k(t) = M_k(t) + S_{k-1}(t).$$  \hspace{1cm} (4.18)

Then, we first show the attractivity of equilibrium. The proof follows from a similar argument in [39, 60].

**Lemma 4.4.2** For the group of $N$ agents (4.3) under the homogeneous condition, there exists $x_i^*$ such that $\lim_{t \to \infty} x_i(t) = x_i^*$, and $x_i^* \in [\min_{j \in \mathcal{V}} x_j(t_0), \max_{j \in \mathcal{V}} x_j(t_0)], \forall i \in \mathcal{V}$.

**Proof:** Since $S_i(t)$ is absolutely continuous for almost everywhere, the derivative of
\( S_m(t) \) is given by

\[
D^+ S_m(t) = D^+ \sum_{i=1}^{m} M_i(t)
\]

\[
= \sum_{k=1}^{m} \sum_{j=1}^{N} \alpha_{ikj}(t)(y_j(t) - y_{ik}(t))
\]

\[
= \sum_{k=1}^{m} \sum_{j=1}^{m} \alpha_{ikj}(t)(y_j(t) - y_{ik}(t)) + \sum_{j=1}^{N} \sum_{j=m+1}^{m} \alpha_{ikj}(t)(y_j(t) - y_{ik}(t))
\]

\[
= \sum_{k=1}^{m} \sum_{j=m+1}^{N} \alpha_{ikj}(t)(y_j(t) - y_{ik}(t)),
\]

(4.19)

which implies \( D^+ S_m(t) \leq 0 \). Therefore, \( S_m(t) \) is nonincreasing function. Moreover, \( S_m(t) \) is bounded below, \( S_m(t) \) converges as \( t \to \infty \). Since \( M_m(t) = S_m(t) - S_{m-1}(t) \), then \( M_m(t) \) converges, too. This implies that every \( M_i(t) \) converges to a limit \( \lim_{t \to \infty} M_i(t) = M_i^\ast \).

Then, from the definition of \( M_i(t) \), each \( x_i(t) \) must converge to one of the values of \( M_i^\ast \).

Moreover, we can easily see from (4.19) that \( D^+ S_1(t) = D^+ M_1(t) \leq 0 \) and \( D^+ S_N(t) = D^+ M_N(t) + D^+ S_{N-1}(t) = 0 \), which implies \( D^+ M_N(t) \geq 0 \) since \( S_{N-1}(t) \leq 0 \). Therefore, we have \( M_i^\ast \in [M_N(t_0), M_1(t_0)] = [\min_{j \in V} x_j(t_0), \max_{j \in V} x_j(t_0)] \).

We next recall the integral graph \( \tilde{G}_{[0, \infty)} \) defined in Chapter 2.1.2. Let \( \tilde{L} \) be the Laplacian of \( \tilde{G}_{[0, \infty)} \). Then, similarly to Lemma 4.3 in [39], we have

**Lemma 4.4.3** \( x^\ast \in \{ x \in \mathbb{R}^N : \text{sat}(x) \in \text{Ker} \tilde{L} \} \).

Then, we present the following result.

**Theorem 4.4.1** Suppose the undirected, time-varying graph \( \tilde{G}(t) \) is integrally connected over \([0, \infty)\), i.e., \( \tilde{G}_{[0, \infty)} \) is connected. Then, the group of \( N \) agents (4.3) under the homogeneous condition achieves the consensus, if and only if \( \frac{1}{N} \left| \sum_{i=1}^{N} x_i(t_0) \right| \leq s \).
Proof: The necessity directly follows from the case of the fixed graph. Therefore, we will prove the sufficiency only.

From Lemma 4.4.2 and Lemma 4.4.3 we know that \( \lim_{t \to \infty} x(t) = x^* \), where \( x^* \) satisfies \( \text{sat}(x^*) \in \text{Ker} \bar{L} \). Moreover, since the integral graph \( \tilde{G}_{(0,\infty)} \) is connected, we have \( \text{Ker} \bar{L} = \text{span}\{1\} \), which implies \( \text{sat}(x^*) \in \text{span}\{1\} \). Then, the necessity and sufficiency directly follow from the case of the fixed graph.

4.4.2 Heterogeneous agents

In this chapter, we consider the heterogeneous condition, that is a general case of the homogeneous condition. In this case, we need the following additional assumption on the graph:

**Assumption 4.4.2** For any pair \((i,j) \in E\), \(\alpha_{ij}(t) \in [\alpha_{\text{min}}, \alpha_{\text{max}}]\).

Moreover, we assume that without loss of generality, the agents are already sorted such that \(s_1 > \cdots > s_N\). Otherwise, by rearranging the order of the agents, we have this form.

Let \(V_k\) be a subset of the node set \(V\) defined by \(V_k := \{1, 2, \ldots, k\}\). Then, we first consider the following lemma whose proof is given in Chapter 4.8.

**Lemma 4.4.4** Suppose the graph \(G(t)\) is integrally connected with Assumption 4.4.2. Then, for any \(x_i(t_0) \in \mathbb{R}, \forall i \in V\), there exists a number \(T \geq t_0\) such that for \(\forall t \geq T\), there holds \(x_i \in (-s_i, s_i), \forall i \in V_{N-1}\). Moreover, for \(\forall i \in V_{N-1}\), we have \(\lim_{t \to \infty} |x_i(t)| \leq s_N\).

We next investigate the attractivity of equilibrium, similarly to the homogeneous case. Let \(M_k(t)\) be the \(k\)-th largest value of the components \(y_i(t)\), that is, we rank \(y_i(t)\) with
descending order for each $t$

$$y_1(t) \geq y_2(t) \geq \cdots \geq y_N(t), \quad (4.20)$$

where $\{i_1, \ldots, i_N\}$ is a permutation of $\{1, \ldots, N\}$, and we define

$$M_k(t) = x_{i_k}(t). \quad (4.21)$$

Let $S_k(t)$ be the sum of the first $k$ largest values of $x_i(t)$, i.e.,

$$S_0(t) = 0, \quad S_k(t) = M_k(t) + S_{k-1}(t). \quad (4.22)$$

Then, similar to Lemma 4.4.2, we can show the existence of limits.

**Lemma 4.4.5** For the group of $N$ agents (4.3) under the heterogeneous condition, there exists $x_i^*$ such that $\lim_{t \to \infty} x_i(t) = x_i^* \forall i \in \mathcal{V}$. Moreover, for $\forall i \in \mathcal{V}_{N-1}$, $x_i^* \in [-s_N, s_N]$.

**Proof:** Since $S_i(t)$ is absolutely continuous for almost everywhere, following the proof of Lemma 4.4.2, we can obtain

$$D^+ S_m(t) = \sum_{k=1}^{m} \sum_{j=m+1}^{N} \alpha_{i_k j}(t)(y_{i_j}(t) - y_{i_k}(t)), \quad (4.23)$$

which implies $D^+ S_m(t) \leq 0$. From Lemma 4.4.4, we know that there exists $T \geq t_0$ such that $x_i(t) \in (-s_{N-1}, s_{N-1})$, $\forall i \in \mathcal{V}_{N-1}$, $\forall t \geq T$. Moreover, the average is invariant, $x_N(t)$ is bounded, $\forall t \geq T$, and $S_m(t)$ is bounded below for $t \geq T$. Consequently, $S_m(t)$ converges as $t \to \infty$, and thus, $M_i(t)$ converges to a limit $M_i^* = \lim_{t \to \infty} M_i(t)$. Then, from the definition of $M_i(t)$, each $x_j(t)$ must converge to one of the values of $M_j^*$. Moreover, from Lemma 4.4.4, we know that $\lim_{t \to \infty} |x_i(t)| \leq s_N$ for all $i \in \mathcal{V}_{N-1}$. Therefore, the equilibrium of $x_i$ for all $i \in \mathcal{V}_{N-1}$ must be within the interval $[-s_N, s_N]$. $\blacksquare$
Lemma 4.4.6 \( x^* \in \{ x \in \mathbb{R}^N : \text{sat}_N(x) \in \text{Ker} \bar{L}, \ |x| \leq s_N \forall i \in \mathcal{V}_{N-1} \} \).

**Proof:** From Lemma 4.4.3 we know that \( y^* \in \text{Ker} \bar{L} \). Since \( \lim_{t \to \infty} |x_i(t)| = |x_i^*| \leq s_N, \forall i \in \mathcal{V}_{N-1} \) from Lemma 4.4.5 \( y_i^* = x_i^* = \text{sat}_N(x_i^*), \forall i \in \mathcal{V}_{N-1}, \) which completes the proof. 

Then, we are now ready to state the following result.

**Theorem 4.4.2** Suppose the undirected, time-varying graph \( G(t) \) is integrally connected over \([0, \infty)\) with Assumption 4.4.1. Then, the group of \( N \) agents (4.3) under the heterogeneous condition achieve the consensus, if and only if \( \frac{1}{N} \sum_{i=1}^{N} x_i(t_0) \leq \min_{i \in \mathcal{V}} \{ s_i \} \).

**Proof:** The necessity directly follows from the case of fixed graph. Therefore, we will prove the sufficiency only.

From Lemma 4.4.5, Lemma 4.4.6 and the fact that \( \text{Ker} \bar{L} = \text{span}\{1\} \), there exists a constant \( C \) such that \( \lim_{t \to \infty} x_i = x_i^* = C \in [-s_N, s_N] \forall i \in \mathcal{V}_{N-1} \), but this does not implies \( \lim_{t \to \infty} x_N(t) = x_N^* = C \). Therefore, we will next show the convergence of \( x_N \) to \( C \). Moreover, since the average value is invariant, we have \( \lim_{t \to \infty} \frac{1}{N} \sum_{i=1}^{N} x_i(t) = \frac{1}{N} \sum_{i=1}^{N} x_i(t_0) = \frac{1}{N}((N - 1)C + x_N^*) \). Therefore, if \( \frac{1}{N} \sum_{i=1}^{N} x_i(t_0) \leq \min_{i \in \mathcal{V}} \{ s_i \} \), then \( x_N^* = C = \frac{1}{N} \sum_{i=1}^{N} x_i(t_0) \) is an unique equilibrium point, which completes the proof.

### 4.4.3 Unachievable Equilibrium

In this chapter, we investigate some properties of unachievable equilibrium for consensus. For simplicity, we assume that \( \frac{1}{N} \sum_{i=1}^{N} x_i(t_0) > \min_{i \in \mathcal{V}} \{ s_i \} \).

For the homogeneous case, as mentioned in the introduction chapter, the set of unachievable equilibrium is defined by \( \Omega_u := \{ x \in \mathbb{R}^N : x \geq s \} \). Then, it is clear from Chapter 4.4.1
that \( \lim_{t \to \infty} x_i(t) = x_i^* \in [s, \max_{i \in V} x_i(t_0)] \). Moreover, the derivative of \( |x_i(t)| \) is given by

\[
D^+ |x_i(t)| \leq \sum_{j=1}^{N} \alpha_{ij}(t)(|y_j(t)| - |y_i(t)|) \\
\leq \sum_{j=1}^{N} \alpha_{ij}(t)(s - |y_i(t)|).
\] (4.24)

Then, for \( |y_i(t)| = s \), \( D^+ |x_i(t)| \leq 0 \), which implies that the set \( O_i := \{x_i : |x_i| \leq s\} \) is a positively invariant set, i.e., if \( x_i(t^*) \in O_i \), then \( x_i(t) \in O_i \) \( \forall t \geq t^* \). Therefore, \( \lim_{t \to \infty} x_i(t) = s, \forall i \in \{i \in V : x_i(t_0) \leq s\} \), and the remaining agents converge to the interval \([s, \max_{i \in V} x_i(t_0)]\).

For the heterogeneous case, according to Chapter 4.4.2, the set of unachievable equilibrium is defined by \( \Omega_u := \{x \in \mathbb{R}^N : x_i = s_N, \forall i \in V_{N-1}, x_N > s_N\} \), which implies, for any \( x_i(t_0) \), \( \lim_{t \to \infty} x_i(t) = s_N, \forall i \in V_{N-1} \), and \( \lim_{t \to \infty} x_N(t) > s_N \). Moreover, the invariance of the average value implies \( \lim_{t \to \infty} \sum_{i=1}^{N} x_i(t) = \lim_{t \to \infty} x_N(t) + (N - 1)s_N = \sum_{i=1}^{N} x_i(t_0) \), which gives \( \lim_{t \to \infty} x_N(t) = \sum_{i=1}^{N} x_i(t_0) - (N - 1)s_N \).

4.5 Extensions

4.5.1 Double-integrator agents

Since many real systems are controlled by the acceleration rather than the velocity, this chapter extends the previous results to the double-integrator modeled agents. Consider the following group of \( N \) double-integrator modeled agents:

\[
\dot{x}_i = v_i \\
\dot{v}_i = u_i, \quad i \in V := \{1, \ldots, N\}.
\] (4.25)
where \( x_i, v_i, u_i \in \mathbb{R} \) are the position (or angle), velocity (or angular velocity), and control input of the agent \( i \), respectively. It was shown that the following consensus algorithm proposed in [12]

\[
    u_i = \sum_{j=1}^{N} \alpha_{ij} ((x_j - x_i) + (v_j - v_i)),
\]

solves the consensus problem for any \( x_i(t_0) \) and \( v_i(t_0) \), i.e.,

\[
    \lim_{t \to \infty} x_i(t) = \frac{1}{N} \sum_{i=1}^{N} x_i(t_0) + t \frac{1}{N} \sum_{i=1}^{N} v_i(t_0),
    \lim_{t \to \infty} v_i(t) = \frac{1}{N} \sum_{i=1}^{N} v_i(t_0), \quad \forall i \in \mathcal{V}.
\]

However, in the presence of the measurement saturations, the consensus may not be reached due to the existence of unachievable equilibrium. In this chapter, we assume that the measurements of the velocities have the homogeneous saturation levels and thus consider the following consensus algorithm:

\[
    u_i = \sum_{j=1}^{N} \alpha_{ij} ((x_j - x_i) + (y_j - y_i)),
    y_i = \text{sat}(v_i).
\]

Then, by extending Theorem 4.3.1 we have the following result:

**Theorem 4.5.1** Suppose the graph is undirected and connected. Then, the group of \( N \) agents (4.25) under the consensus algorithm (4.28) achieves the consensus, i.e.,

\[
    \lim_{t \to \infty} (x_i - x_j) = 0 \quad \text{and} \quad \lim_{t \to \infty} (v_i - v_j) = 0, \quad \forall i, j \in \mathcal{V}, \text{ if and only if } \frac{1}{N} \sum_{i=1}^{N} v_i(t_0) \leq s.
\]

**Proof:** Since the average of all velocities is invariant, the necessity directly follows from Theorem 4.3.1 Therefore, we will prove the sufficiency only.
Consider the following Lyapunov function candidate:

$$V = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_{ij} (x_i - x_j)^2 + \sum_{i=1}^{N} v_i^2. \quad (4.29)$$

Then, the time derivative of $V$ is given by

$$\dot{V} = \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_{ij} (x_i - x_j)(\dot{x}_i - \dot{x}_j) + 2 \sum_{i=1}^{N} v_i \dot{v}_i$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_{ij} (x_i - x_j)(v_i - v_j) + 2 \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_{ij} v_i (x_j - x_i + y_j - y_i). \quad (4.30)$$

Note that, by applying Lemma 4.3.2, we have

$$2 \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_{ij} v_i (x_j - x_i + y_j - y_i) = - \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_{ij} (v_i - v_j)(x_i - x_j + y_i - y_j), \quad (4.31)$$

which gives

$$\dot{V} = - \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_{ij} (v_i - v_j)(y_i - y_j). \quad (4.32)$$

Since the saturation function satisfies the incremental passive condition \[61\], i.e.,

$$(v_i - v_j)(\text{sat}(v_i) - \text{sat}(v_j)) \geq 0, \text{ for any } i, j \in \mathcal{V}, \quad (4.33)$$

we have $\dot{V} \leq 0$. Let $\mathcal{M} := \{(x, v) \in \mathbb{R}^{2N} : \dot{V} = 0\}$. Then, $\dot{V} \equiv 0$ implies that either $(v_i - v_j) \equiv 0$ or $(\text{sat}(v_i) - \text{sat}(v_j)) \equiv 0, \forall i, j \in \mathcal{V}$. It is clear from the proof of Theorem 4.3.1 that if $\frac{1}{N} \left| \sum_{i=1}^{N} v_i(t_0) \right| \leq s$, then $(\text{sat}(v_i) - \text{sat}(v_j)) \equiv 0, \forall i, j \in \mathcal{V}$, only when $(v_i - v_j) \equiv 0, \forall i, j \in \mathcal{V}$. Moreover, $(v_i - v_j) \equiv 0, \forall i, j \in \mathcal{V}$, implies $(\dot{v}_i - \dot{v}_j) \equiv 0, \forall i, j \in \mathcal{V}$, in an invariant set within $\mathcal{M}$, which gives that $\dot{v} \in \text{span}\{1\}$. Note that the average of all velocities is invariant, i.e., $1^T \dot{v} = 0$, and thus, $\dot{v}$ is orthogonal to 1. Therefore, we can conclude that $\dot{v} \equiv 0$, and thus, from (4.28) and the fact that $(v_i - v_j) \equiv 0$, it follows that
\[ \dot{v}_i = - \sum_{j=1}^{N} \alpha_{ij} x_{ij} \equiv 0. \] As a result, we have \[ \sum_{i=1}^{N} x_i \sum_{j=1}^{N} \alpha_{ij} x_{ij} \equiv 0, \] which implies from Lemma 4.3.2 that \( \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_{ij} (x_i - x_j)^2 \equiv 0. \) Since the graph is connected, we can conclude that \( (x_i - x_j) \equiv 0, \forall i, j \in \mathcal{V}. \)

In summary, we have shown that \( \dot{V} \leq 0 \) and \( \dot{V} \equiv 0 \) only when \((x_i - x_j) \equiv 0\) and \((v_i - v_j) \equiv 0, \forall i, j \in \mathcal{V}. \) Therefore, according to Lasalle Invariance Principle, we have \[ \lim_{t \to \infty} (x_i(t) - x_j(t)) = 0 \] and \[ \lim_{t \to \infty} (v_i(t) - v_j(t)) = 0, \forall i, j \in \mathcal{V}, \] which completes the proof.

### 4.5.2 Directed graph

In this chapter, we consider the single-integrator modeled agents as in (4.3) with a directed graph. Let \( p = [p_1, \ldots, p_N]^T \) be the left eigenvector of its Laplacian matrix \( L \) associated with eigenvalue \( \lambda_1 = 0, \) and \( \sum_{i=1}^{N} p_i = 1. \) Note that \( p \) is positive [62], and it is clear that the weighted average of all agents’ states defined by \( \sum_{i=1}^{N} p_i x_i(t) \) is invariant. Then, we have the following lemma:

**Lemma 4.5.1** [62] For a strongly connected, directed graph, and any \( y_i \in \mathbb{R}, i = 1, \ldots, N, \) we have

\[ 2 \sum_{i=1}^{N} \sum_{j=1}^{N} p_i \alpha_{ij} (y_i - y_j) = \sum_{i=1}^{N} \sum_{j=1}^{N} p_i \alpha_{ij} (y_i - y_j)^2. \] (4.34)

**Theorem 4.5.2** Suppose the graph is directed and strongly connected. Then, the group of \( N \) agents (4.3) achieves the consensus, if and only if \( |\sum_{i=1}^{N} p_i x_i(t_0)| \leq \min_{i \in \mathcal{V}} \{s_i\}. \)

**Proof:** Since the weighted average of all agents’ states is invariant, the necessity can be proved similar to the case of the undirected graph. Therefore, we will prove the sufficiency only.
Let \( x^* = \sum_{i=1}^{N} p_i x_i(t_0) \), and assume that \( |x^*| \leq \min_{i \in V} s_i \). Consider the following Lyapunov function candidate:

\[
V = 2 \sum_{i=1}^{N} p_i \int_{x^*}^{x_i} \text{sat}_i(\tau) d\tau.
\] (4.35)

Since \( \sum_{i=1}^{N} p_i = 1 \) and \( x^* = \sum_{i=1}^{N} p_i x_i \), we have

\[
\sum_{i=1}^{N} p_i \int_{x^*}^{x_i} x^* d\tau = x^* \sum_{i=1}^{N} p_i x_i - \sum_{i=1}^{N} p_i x_i^2 = 0.
\] (4.36)

Hence, by subtracting \( 2 \sum_{i=1}^{N} p_i \int_{x^*}^{x_i} x^* d\tau \) from the both side of (4.35), and following the procedure as in (4.11), we can show \( V \geq 0 \). We next consider the time derivative of \( V \) given by

\[
\dot{V} = 2 \sum_{i=1}^{N} p_i \text{sat}_i(x_i) \dot{x}_i
\]

\[
= 2 \sum_{i=1}^{N} p_i y_i \sum_{j=1}^{N} \alpha_{ij} (y_j - y_i).
\] (4.37)

Then, from Lemma 4.5.1 it follows that

\[
\dot{V} = - \sum_{i=1}^{N} \sum_{j=1}^{N} p_i \alpha_{ij} (y_i - y_j)^2,
\] (4.38)

which implies \( \dot{V} \leq 0 \). Let \( \mathcal{M} := \{ x \in \mathbb{R}^N : \dot{V} = 0 \} \). Then, since the graph is strongly connected, \( \dot{V} \equiv 0 \) implies that \( (y_i - y_j) \equiv (\text{sat}_i(x_i) - \text{sat}_j(x_j)) \equiv 0, \forall i,j \in V \). Then, similar to the proof of Theorem 4.3.1 we can prove that \( \dot{V} \equiv 0 \) only when \( (x_i - x_j) \equiv 0, \forall i,j \in V \). Therefore, applying LaSalle Invariance Principle gives \( \lim_{t \to \infty} (x_i(t) - x_j(t)) = 0, \forall i,j \in V \), which completes the proof.

\[\blacksquare\]
4.6 Simulation results

4.6.1 Fixed Graph

We consider a group of 50 agents whose topology is fixed, undirected and connected.

We first consider the homogeneous agents with $s = 1$. The initial conditions are uniformly distributed on the interval $[-10, 10]$. Fig. 4.1 shows that the simulation results with the average values are (a) 0.9402 and (b) 1.4226. Then, the case (a) satisfies the condition in Theorem 4.3.1 and thus the agents achieve the consensus. However, the case (b) does not satisfy the condition in Theorem 4.3.1 and consequently, the consensus is not reached.

We next consider the heterogeneous agents. We choose the saturation levels on the interval $s_i \in [1, 7], \forall i \in \mathcal{V}$ and $\min_{i \in \mathcal{V}} \{s_i\} = 1$. Fig. 4.2 shows that the simulation results with the average values are (a) 0.9368 and (b) 1.3462. From Theorem 4.3.1, it is clear that the case (a) achieves the consensus, but the case (b) is not. Moreover, in the case (b), there are 2 agents whose saturation level is 1. As we discussed in Chapter 4.4.3, the agents except for 2 agents, whose saturation levels are 1, converge to 1.

4.6.2 Time-varying Graph

We consider a group of 4 agents whose graph topology is time-varying. We assume that the network is changed between three graphs in Fig. 4.3 which are described by
Figure 4.1: Homogeneous agents with fixed graph.

(a) $\frac{1}{N} \sum_{i=1}^{N} x_i(0) = 0.9402$

(b) $\frac{1}{N} \sum_{i=1}^{N} x_i(0) = 1.4226$
Figure 4.2: Heterogeneous agents with fixed graph.

(a) $\frac{1}{N} \sum_{i=1}^{N} x_i(0) = 0.9368$

(b) $\frac{1}{N} \sum_{i=1}^{N} x_i(0) = 1.3462$
\[ A_1(t) = (3 + \sin(t)) \ast \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2(t) = (2 - \cos(t)) \ast \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \]

\[ A_3(t) = (1.5 - \sin(t)) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \] (4.39)

over a sequence \(0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots\) with \(t_{k+1} - t_k = 10(s)\). Then, for \(k = 0, 1, 2, \ldots\),

\[ A(t) = \begin{cases} 
A_1(t) & \text{if } t \in [t_k, t_k + \delta_1), \\
A_1(t) & \text{if } t \in [t_k + \delta_1, t_k + \delta_2), \\
A_1(t) & \text{if } t \in [t_k + \delta_2, t_{k+1}), 
\end{cases} \]

where \(\delta_1 = 3\) and \(\delta_2 = 6\). Note that this network is disconnected all the time.

Then, we first consider the homogeneous agents with \(s = 1\). Fig. 4.4 shows that the simulation results with the average values are (a) \(-0.75\) and (b) \(1.25\). Since the graph is integrally connected over \([0, \infty)\), it is clear that from Theorem 4.4.1, the case (a) achieves the consensus, but the case (b) is not.

We next consider the heterogeneous agents with \(s_i = i \forall i \in \mathcal{V} := \{1, 2, 3, 4\}\). With the same initial conditions as used in the homogeneous case, the simulation results is given in Fig. 4.5. From Theorem 4.4.2, it is clear that the case (a) achieves the consensus, but the case
(a) $t \in [t_k, t_k + \delta_1)$

(b) $t \in [t_k + \delta_1, t_k + \delta_2)$

(c) $t \in [t_k + \delta_2, t_{k+1})$

Figure 4.3: Three graphs in Chapter 4.6.2

(b) is not. Moreover, from Chapter 4.4.3, the agents except for 1 agent converge to 1.

4.6.3 Double-Integrator

We consider a group of 10 double-integrator modeled agents whose topology is fixed, undirected and connected, and the homogeneous saturation level with $s = 1$. The initial conditions are uniformly distributed on the interval $[-10, 10]$. Fig. 4.6 and Fig. 4.7 show the simulation results with the average of all velocities are (a) $-0.85$ and (b) $1.85$, respectively. As we can see from the simulation results, the agents achieve the consensus for the case (a), but not for the case (b).

4.6.4 Directed graph

We consider a group of 6 agents whose graph topology is fixed, directed and strongly connected as depicted in Fig. 4.8 and the homogeneous saturation level with $s = 1$. From the Laplacian matrix as in Fig. 4.8(b), its left eigenvector is given by

$$p = [0.0678, 0.0339, 0.2373, 0.1186, 0.2712, 0.2712]^T.$$ (4.40)
Figure 4.4: Homogeneous agents with time-varying graph.

(a) \( \frac{1}{N} \sum_{i=1}^{N} x_i(0) = -0.75 \)

(b) \( \frac{1}{N} \sum_{i=1}^{N} x_i(0) = 1.25 \)
Figure 4.5: Heterogeneous agents with time-varying graph.

(a) $\frac{1}{N} \sum_{i=1}^{N} x_i(0) = -0.75$

(b) $\frac{1}{N} \sum_{i=1}^{N} x_i(0) = 1.25$
Figure 4.6: Double-integrators with $\frac{1}{N} \sum_{i=1}^{N} v_i(0) = -0.85$.

Fig. 4.9 shows simulation results with the weighted averages, $\sum_{i=1}^{N} p_i x_i(0)$, as (a) 0.3455 and (b) $-1.8450$, respectively. From Theorem 4.5.2 it is clear that the case (a) achieves the consensus, but the case (b) is not.

4.7 Conclusion

In this chapter, we have studied the consensus problem with output saturations. Due to the existence of unachievable equilibrium for the consensus, the agents may not achieve the consensus in the presence of output saturations. Therefore, we have investigated the condi-
tions for achieving the consensus. We have discussed both homogeneous and heterogeneous saturation levels, and fixed and time-varying graphs. To find the consensus conditions, we have analyzed the attractivity of equilibrium. Then, by investigating the equilibrium, the necessary and sufficient conditions for achieving the consensus have been derived.

4.8 Proof of Lemma 4.4.4

The proof of Lemma 4.4.4 is outlined as follows. We first show that for any \( x_i(t_0) \in \mathbb{R} \) \( \forall i \in \mathcal{V} \), \( x_1(t) \) will converge to its linear region in finite time and remains in it, that is \( \exists T_1 \geq t_0 \) such that \( |x_1(t)| \leq s_1, \forall t \geq T_1 \). We next show that \( |x_1(t)| \) for \( t \geq T_1 \) will converge to \( s_2 \).
faster than exponential. By repeating this process for $\forall i \in V_{N-1}$, we will prove Lemma 4.4.4. To complete this process, we need the following lemmas.

**Lemma 4.8.1** If $x_i(t^*) \in [-s_k, s_k], \forall i \in V_k := \{1, 2, ..., k\}, t^* \geq t_0$, then $x_i(t) \in [-s_k, s_k], \forall i \in V_k and \forall t \geq t^*$.

**Proof:** Let $V_M(x(t)) = \max_{i \in V_k} \{x_i(t)\}$. Then, we will show that $D^+ V_M(x(t)) \leq 0$ when $V_M(x(t)) = s_k$. Let $S(t) = \{i \in V_k : x_i(t) = \max_{i \in V_k} \{x_i(t)\}\}$ be the index set where
the maximum is reached at $t$, and consider the upper Dini derivative of $V_M$ as follows:

$$D^+ V_M(x(t)) = \max_{i \in S} \dot{x}_i = \max_{i \in S} \sum_{j=1}^{N} \alpha_{ij}(t)(y_j - y_i).$$

(4.41)
Then, for $V_M = s_k$ and $t \geq t^*$, it follows that

$$D^+ V_M(x(t)) = \max_{i \in S} \left( \sum_{j=1}^{k} \alpha_{ij}(t)(x_j - x_i) + \sum_{k+1}^{N} \alpha_{ij}(t)(y_j - x_i) \right) \leq 0.$$  \hspace{1cm} (4.42)

We next define $V_m(x(t)) = \min_{i \in V_k} \{x_i(t)\}$. Then, we can similarly show that for $V_m = -s_k$,

$$D^+ V_m \geq 0,$$

which complete the proof. ■

We next consider the following group of $N$ agents:

$$\dot{x}_i = \sum_{j=1}^{N} \alpha_{ij}(t)(x_j - x_i) - d_i(t)x_i,$$  \hspace{1cm} (4.43)

where $d_i(t)$ is continuous except for a set with measure zero, and satisfies $d_i(t) \geq 0$, $\forall t \geq t_0$ $\forall i \in V$.

**Definition 4.8.1** The agents (4.43) is said to be exponentially converges to its equilibrium $x^*$ with respect to $k$ if there exist two constants $\Delta, \delta > 0$ such that $||x(t_k) - x^*|| \leq \Delta e^{-\delta k}||x(t_0) - x^*||$

**Lemma 4.8.2** Suppose that the graph $G(t)$ is integrally connected with Assumption 4.4.2. Then,

1) the agents (4.43) exponentially achieve the consensus with respect to $k$.

2) if there exists at least one agent such that $\int_{t_0}^{\infty} d_i(t) = \infty$, then the equilibrium point is given by the origin.

**Proof**: Since $d_i(t) \geq 0$, $\forall t \geq t_0$, the proof of the condition 1) directly follows from Theorem 5.2 in [41]. We next prove the condition 2). Let $S(t) = \sum_{i=1}^{N} M_i(t)$, and then the
derivative of $S(t)$ is given by

$$D^+ S(t) = D^+ \sum_{i=1}^{N} M_i(t)$$

$$= N \sum_{i=1}^{N} \sum_{j=1}^{N} (\alpha_{ij}(t)(x_j(t) - x_i(t)) - d_i(t)x_i(t))$$

$$= - \sum_{i=1}^{N} d_i(t)x_i(t),$$

(4.44)

and thus the solution $S(t)$ is

$$S(t) = S(t_0) - \int_{t_0}^{t} \sum_{i=1}^{N} d_i(\tau)x_i(\tau)d\tau. \quad (4.45)$$

From the condition 1), we know that the agents (4.43) achieve the consensus, and thus $S(t)$ converges to some $S^*$. Therefore, it follows that

$$\int_{t_0}^{\infty} \sum_{i=1}^{N} d_i(\tau)|x_i(\tau)|d\tau = |S(t_0) - S^*| < \infty. \quad (4.46)$$

Then, if there exists at least one agent such that $\int_{t_0}^{\infty} d_i(\tau)d\tau = \infty$, $\int_{t_0}^{\infty} \sum_{i=1}^{N} d_i(\tau)|x_i(\tau)|d\tau < \infty$ implies that $|x_i(t)|$ converges must be 0, which complete the proof. \hfill \blacksquare

Then, now we are going to prove Lemma 4.4.4.

**Proof of Lemma 4.4.4**

Step 1. As mentioned above, we will first show that, for any $x_i(t_0) \in \mathbb{R} \ \forall i \in \mathcal{V}$, $x_1(t)$ will enter the interval $(-s_1, s_1)$ in finite time, and remains in it.

Consider the time derivative of $|x_1(t)|$ as follows:

$$D^+ |x_1(t)| \leq \sum_{j=1}^{N} \alpha_{1j}(t)(|y_j(t)| - |y_1(t)|)$$

$$\leq \sum_{j=1}^{N} \alpha_{1j}(t)(s_2 - |y_1(t)|). \quad (4.47)$$
Then, the solution is given by

$$|x_1(t)| \leq |x_1(t_0)| + \int_{t_0}^{t} \sum_{j=1}^{N} \alpha_{1j}(\tau)(s_2 - y_1(\tau))d\tau.$$  \hspace{1cm} (4.48)

If $|x_1(t)| \geq s_1$, $\forall t \geq t_0$, it follows that

$$|x_1(t)| \leq |x_1(t_0)| + \int_{t_0}^{t} \sum_{j=1}^{N} \alpha_{1j}(\tau)(s_2 - s_1)d\tau$$

$$\leq |x_1(t_0)| - s_{1,2} \int_{t_0}^{t} \sum_{j=1}^{N} \alpha_{1j}(\tau)d\tau,$$  \hspace{1cm} (4.49)

where $s_{i,j} = s_j - s_i$. Since the graph $G(t)$ is integrally connected, i.e., $\int_{t_0}^{\infty} \sum_{j=1}^{N} \alpha_{1j}(\tau)d\tau = \infty$, it follows that $\lim_{t \to \infty} |x_1(t)| = -\infty$, which contradicts $|x_1(t)| \geq 0$, $\forall t \geq t_0$. Moreover, from Lemma 4.8.1, we can conclude that there exists $T > 0$ such that for $\forall t \geq T$ and any $|x_1(t_0)| \geq s_1$, it holds $x_1(t) \in (-s_1, s_1)$. Moreover, since the consensus algorithm is bounded and the average value is invariant, the remaining states remain bounded for any finite time (see, [63]).

Step $p$, $p = 2, \ldots, N - 1$. In this step, we will show that, for any $x_p(t_0) \in \mathbb{R}$ and $p = 2, \ldots, N - 1$, $x_p(t)$ will enter the interval $(-s_p, s_p)$ in finite time, and remains in it.

In the previous step, we have shown that for $\forall i \in \mathcal{V}_{p-1}$, $x_i$ will enters and remains in the interval $(-s_i, s_i)$ in finite time. Thus, the resulting dynamics of agent $i$ for $i \in \mathcal{V}_{p-1}$ is given by

$$\dot{x}_i(t) = \sum_{j=1}^{p-1} \alpha_{ij}(t)(x_j(t) - x_i(t)) + \sum_{j=p}^{N} \alpha_{ij}(t)(y_j(t) - x_i(t)).$$  \hspace{1cm} (4.50)
We next consider the upper Dini derivative of \(|x_i(t)|\) for \(i \in \mathcal{V}_{p-1}\) as follows:

\[
D^+|x_i(t)| \leq \sum_{j=1}^{p-1} \alpha_{ij}(t)(|x_j(t)| - |x_i(t)|) + \sum_{j=p}^{N} \alpha_{ij}(t)(|y_j(t)| - |x_i(t)|)
\]

\[
\leq \sum_{j=1}^{p-1} \alpha_{ij}(t)(|x_j(t)| - |x_i(t)|) + \sum_{j=p}^{N} \alpha_{ij}(t)(s_p - |x_i(t)|). \tag{4.51}
\]

Let \(p(x, t) = [|x_1(t)|, ..., |x_{p-1}(t)|]^{T}\), and \(L_{p-1}(t) \in \mathbb{R}^{p-1 \times p-1}\) be a Laplacian of the subgraph \(\mathcal{G}_{p-1}(t) = (\mathcal{V}_{p-1}, \mathcal{E}_{p-1}(t), \mathcal{A}_{p-1}(t)) \subset \mathcal{G}(t)\), and define a diagonal matrix \(D_{p-1}(t) = \text{diag}\left(\sum_{j=p}^{N} \alpha_{1j}(t), ..., \sum_{j=p}^{N} \alpha_{p-1j}(t)\right)\). Then, we have

\[
D^+p(x, t) \leq -(L_{p-1}(t) + D_{p-1}(t)) p(x, t) + D_{p-1}(t)s_p 1. \tag{4.52}
\]

We next consider the following comparison system for \(i \in \mathcal{V}_{p-1}\):

\[
\dot{z}(t) = -(L_{p-1}(t) + D_{p-1}(t)) z(t) + D_{p-1}(t)s_p 1, \quad z \in \mathbb{R}_+^{p-1}. \tag{4.53}
\]

By denoting the error vector \(\bar{z} = z - s_p 1\), we have

\[
\dot{\bar{z}}(t) = -(L_{p-1}(t) + D_{p-1}(t)) \bar{z}(t). \tag{4.54}
\]

Since the graph \(\mathcal{G}(t)\) is integrally connected over \([0, \infty)\), without loss of generality, we assume that there are \(m\) integrally connected subgraph in \(\mathcal{G}_{p-1}(t)\) over \([0, \infty)\), where \(p - 1 \geq m \geq 1\). Then, by rearranging the order of the nodes, the Laplacian matrix \(L_{p-1}(t)\) can be written in the block matrix form as \(L_{p-1}(t) = \text{blkdiag}\left(L_{p-1}^1(t), ..., L_{p-1}^m(t)\right)\), where \(L_{p-1}^i(t)\) for \(i = 1, ..., m\) is the Laplacian matrix of the corresponding integrally connected subgraph of \(\mathcal{G}_{p-1}(t)\). We can similarly rewrite the diagonal matrix \(D_{p-1}(t)\) as \(D_{p-1}(t) = \text{blkdiag}\left(D_{p-1}^1(t), ..., D_{p-1}^m(t)\right)\) with \(D_{p-1}^i(t) = \text{diag}\left(d_{1i}(t), ..., d_{mi}(t)\right)\). Since the graph \(\mathcal{G}(t)\) is integrally connected over \([0, \infty)\), there exists at least one element \(q \in [1, ..., m]\)
for each $i = 1, \ldots, m$ such that $\int_{t_0}^{\infty} d_i^q(t) dt = \infty$. Then, according to Lemma 4.8.2, $\bar{z}(t)$ converges exponentially fast to the origin with respect to $k$, that implies $z_i(t)$ converges exponentially fast to $s_p$ with respect to $k$. Finally, according to the comparison lemma, we can conclude that, for any $|x_i(t_0)| \geq s_p$, $i \in \mathcal{V}_{p-1}$, $x_i$ will enters the interval $[-s_p, s_p]$ faster than exponential with respect to $k$, that is, there exist two constants $\Delta, \delta > 0$ such that for $i \in \mathcal{V}_{p-1}$ and $t \in [t_{k-1}, t_k)$,

$$|x_i(t)| \leq s_p + \Delta(t), \quad (4.55)$$

where $\Delta(t) = \Delta e^{-\delta k}$.

To complete the proof of Step $p$, we will next prove that for any $|x_p(t_k)| \geq s_p$, $x_p$ will converges to the interval $(-s_p, s_p)$ in finite time. We consider the following two cases depending on the graph topology of $\hat{G}_{[0, \infty)}$:

1) $\exists j \in [p + 1, \ldots, N]$ such that $(p, j) \in \mathcal{E}$.

Consider $|x_p(t)|$ and its upper Dini derivative as follows:

$$D^+|x_p(t)| = \sum_{j=1}^{p-1} \alpha_{pj}(t)(|x_j(t)| - |y_p(t)|) + \sum_{j=p}^{N} \alpha_{pj}(t)(|y_j(t)| - |y_p(t)|) \leq \sum_{j=1}^{p-1} \alpha_{pj}(t)(s_p + \Delta(t) - |y_p(t)|) + \sum_{j=p}^{N} \alpha_{pj}(t)(s_{p+1} - |y_p(t)|). \quad (4.56)$$

We next assume that $|x_p(t)| \geq s_p$, $\forall t \geq t_k$. Then, we have

$$|x_p(t)| \leq |x_p(t_k)| + \int_{t_k}^{t} \left( \sum_{j=1}^{p-1} \alpha_{pj}(\tau) \Delta(\tau) - \sum_{j=p}^{N} \alpha_{pj}(\tau) s_{p,p+1} \right) d\tau. \quad (4.57)$$

Since $\alpha_{ij}(t)$ is upper-and lower-bounded from Assumption 4.4.2 and continuous over each time interval, and $\lim_{t \to \infty} \Delta(t) = 0$, it follows that $\lim_{t \to \infty} |x_p(t)| = -\infty$, which is a contradiction. Therefore, for any $|x_p(t_0)| \geq s_p$, there exists $T > 0$ such that it holds $x_p(t) \in (-s_p, s_p), \forall t \geq T$. 

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2) for $\forall j \in [p + 1, \ldots, N]$, $(p, j) \notin \bar{E}$.

In this case, there is an agent $i \in V_{p-1}$ such that $(i, j) \in \bar{E}$, $j \in [p + 1, \ldots, N]$. Then, we consider the agent $i, i \in V_{p-1}$, and its upper Dini derivative as follows:

$$D^+|x_i(t)| \leq \sum_{j=1}^{p-1} \alpha_{ij}(t)(|x_j(t)| - |x_i(t)|) + \sum_{j=p}^{N} \alpha_{ij}(t)(|y_j(t)| - |x_i(t)|).$$

(4.58)

We assume that $|x_i(t)| \geq s_p, \forall t \geq t_k$. Then, we have

$$D^+|x_i(t)| \leq \sum_{j=1}^{p-1} \alpha_{ij}(t)\Delta(t) - \sum_{j=p+1}^{N} \alpha_{ij}(t)s_{p, p+1},$$

(4.59)

which gives

$$|x_i(t)| \leq |x_i(t_k)| + \int_{t_k}^{t} \left( \sum_{j=1}^{p-1} \alpha_{ij}(\tau)\Delta(\tau) - \sum_{j=p+1}^{N} \alpha_{ij}(\tau)s_{p, p+1} \right) d\tau$$

(4.60)

Then, following case 1), we can conclude that there exists $T' > 0$ such that it holds $x_i(t) \in (-s_p, s_p)$, $\forall t \geq T'$. Repeating this argument for every $i \in V_p$, we can conclude that since $\bar{G}_{[0, \infty)}$ is connected, there exists $T \geq T' \geq 0$ such that for any $x_i(t_0) \geq s_p$, $\forall i \in V_p$, it holds $x_i(t) \in (-s_p, s_p)$ $\forall t \geq T$.

Step N. We will show that, for any $x_N(t_0) \in \mathbb{R}$ and $|x_i(t)| \leq s_{N-1}, \forall i \in V_{N-1}, \forall t \geq T$, we have $\lim_{t \to \infty} |x_i| \leq s_N, \forall i \in V_{N-1}$.

Since we have shown in Step 1-to-(N-1) that $|x_i(t)| \leq s_{N-1}, \forall i \in V_{N-1}, \forall t \geq T$, we assume that $|x_i(t_0)| \leq s_{N-1}, \forall i \in V_{N-1}$. Then, for $i \in V_{N-1}$, we have

$$\dot{x}_i(t) = \sum_{j=1}^{N-1} \alpha_{ij}(t)(x_j(t) - x_i(t)) + a_{iN}(t)(y_N(t) - x_i(t)).$$

(4.61)

Then, with the same argumentation as above, we have $\lim_{t \to \infty} |x_i(t)| \leq s_N, \forall i \in V_{N-1}$, which complete the proof.
Chapter 5

Conclusion

5.1 Summary of results

In this dissertation, we have analyzed the consensus of multi-agent systems with saturation nonlinearities. The results can be summarized as follows:

In the presence of the saturation constraints in the interconnection states, we provide new analysis technique for the consensus problem under interconnection constraints by using the state saturation functions. Then, by utilizing the edge dynamics, we provide the sufficient conditions for achieving the consensus. Compared to the existing results, we consider general linear systems and general constraints. Moreover, we extend the analysis technique to the synthesis problem under both the communication ranges limitations and the input constraints. Finally, we show that the proposed technique can be applied to analyze the balancing problem for storage devices under flow constraints.

In the presence of the output saturations, we provide analysis results for the consensus problem. We first consider both fixed and time-varying graphs, and both homogeneous and heterogeneous saturation levels. For each case, we investigate the attractivity with respect to the equilibria, and some behavior properties of each agent. Then, we provide the necessary and sufficient conditions for achieving the consensus under output saturations by investigating conditions for the achievable equilibrium. Moreover, we extend the results to the cases
of double-integrator modeled agents, and the directed graph.

5.2 Possible future research works

Therefore some issues, not addressed in this dissertation: 1) As we addressed in Chapter 1, there are several constraints in real applications. Although this dissertation dealt with the consensus problem under saturation constraints, it is worth to analyze the consensus problem under simultaneous constraints, such as both the interconnection and the output saturations.

2) in Chapter 3.5.2, we have dealt with the load balancing problem. It was assume the supply is given for the storage devices. It would be worthwhile to design the control or coordination algorithms for real applications.

3) in Chapter 4.5.2, we have dealt with the fixed and directed graph. Due to the invariance of the weighted average $\sum_{i=1}^{N} p_i x_i(t)$, we have proved the consensus by extending the result of the undirected graph. However, for the time-varying directed graph, the weighted average is not invariant, and this problem appears quite challenging in a technical sense.

4) the analysis of Chapter 4 can be applied to any bounded nonlinearities, which are strictly increasing within the bounds. However, the nonlinearities should be componentwise with respect to the state vector. It would be worthwhile to extend the results of this chapter to general multi-dimensional systems for real applications.
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